# Supplement Chapter 4

# Implied Variance, Volatility Smile, and CEV Option Pricing Model

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1. **Introduction**

Implied variance is very important information in financial derivative. For example, the VIX, which is traded in future exchange, is implied variance of S&P 500 index option. There are two approaches to estimate implied variance, the non-closed-form solution method, which needs to rely upon complicated computer program to solve it, and the linearized approximation method.

In section 2, we will discuss an Exact Closed-form Solution for the Implied Standard Deviation, which use Tailor Expansion Method to linearize option pricing model. In section 3, we will discuss the Implied Standard Deviation formula under a single call option. Excel programs are also presented in this section for calculating implied standard deviation. In section 4, we will discuss formulas for Implied Standard Deviation under Different Exercise Prices. In section 5, we will discuss the volatility smile. In section 6, we will briefly discuss how MATLAB program can be used to calculate implied variance for both the Black Scholes model and constant elasticity volatility (CEV) of option pricing model. Finally, in section 7, we summarize the results of this Chapter. In Appendix A we will discuss Corrado and Miller’s linearized method. Appendix B discusses the MATLAB approach to estimate implied variance. Appendix C discusses Noncentral χ2 and the Option Pricing Model and

# 2. An Exact Closed-form Solution for the Implied Standard Deviation: A Special Case

2.1 Overview

Black and Scholes’ call option formula can be defined as

,(4.1)

where C = value of option; *S* = current price of stock; *r* = continuously compounded annual risk-free interest rate; *T* = time in years to expiration date; *E* = exercise price; *e* = 2.71828 is a constant; and = probability that a standard normal random variable is less than or equal to , with

, (4.2a)

and

, (4.2b)

where  = variance of annual rate of return on the stock, continuously compounded; and logarithms are to base *e.*.

d1 as defined in (4.2a) can be rewritten as

Similarly

Substituting (4.3a) and (4.3b) into equation (4.1), we can obtain

C = S[N**(+** )] - K[N( **-**)] , (4.4)

where, C is the call premium, S is the underlying asset value, the present value of the exercise price K is equal to Ee-rT. E is the exercise price, r is the instantaneous risk-free rate, T is the time to the maturity, σ is the standard deviation of the underlying asset rate of return on annual basis, and N(x) is the standard cumulative normal distribution function up to x.

Under the condition that the stock price equals the present value of the exercise price (i.e., S=K=Ee-rT ), Brenner and Subrahmanyam (1988) derive the following simple formula to

[[1]](#footnote-1)compute the implied standard deviation.[[2]](#footnote-2)

. (4.5)

This formula is derived by applying Taylor series expansion at zero in the B/S model up to the first-order term, and the higher order terms are ignored. Since the higher order terms are omitted, equation (4.5) is an approximation rather than an exact solution. The accuracy of the ISD calculated by equation (4.5) depends not only on the deviation of the stock price S from the present value of exercise price K but also on the magnitude of the standard deviation σ.

However, when the underlying asset price equals the present value of the exercise price, the B/S model becomes[[3]](#endnote-1)

 (4.6)

Thus, an exact closed-form solution for the ISD[[4]](#endnote-2) is

 (4.7)

Since the cumulative normal distribution is a monotonic increasing function, its inverse cumulative normal functions in equation (4.7) must exist and be unique. Therefore, equation (4.7) must be an exact and unique closed-form solution for the ISD if the present value of the exercise price equals the underlying asset price. It is noteworthy that the Taylor expansion plays no role in deriving equation (4.7) and there is no truncation of the remainder terms. Hence, when the assumption S=K=Ee-rT holds, using equation (4.7) to calculate the ISD results in the true ISD and as there is no estimation error. Previous formulas are approximate because they are derived [[5]](#footnote-3)by truncating the remainder terms in the Taylor formula. The estimate error always exists. Equation (4.7) is the best formula for ISD and dominates all other formulas if the S=K**.**

-(N(****))] = -S + 2 SN(****), solve the ISD yields equation (4.23).

Equation (4.7) is useful to calculate the exact ISD in this special case because the popularity of the Excel, in which the inverse standard cumulative normal function is imbedded.[[6]](#endnote-3)

 Given equation (4.6), the sensitivity of the call option of the exercise price and the hedge ratio at the S=K are determined by

, (4.8)

. (4.9)

Equation (4.8) is the sensitivity of the call option value from the striking price at K=Ee-rT=S Equation (4.9) is the hedge ratio (delta) at the underlying asset price S equals the present value K of exercise price. Equation (4.8) shows that the ∂C/∂K must be greater than -1/2, while equation (4.9) demonstrates that the delta must be greater than 1/2. Equations (4.8) and (4.9) provide a closed-form and exact solutions for the ∂C/∂K and delta. Given the closed-form and exact solution of the ISD in (4.7), it is possible to find all of the closed-form and exact solutions for all of the Greeks of options at S=K. The following theorem summarizes the results of equations (4.7)-(4.9).

*Theorem: Under the Black and Scholes call option pricing model, if underlying asset price S equals to the present value K of exercise price, then there is an exact closed-form solution for the ISD and is given by (4.7). Further,* ∂C/∂*K and the hedge ratio at S=K are given by (4.8) and (4.9) respectively*.

**3. An Implied Standard Deviation Formula under a Single Call Option**

Following Ang et.al (2013), this section derives a formula to estimate the ISD by applying the Taylor series expansion on a single call option. We show that the improved formula for ISD derived by Corrado and Miller (1996)(CM) can be improved further without any replacements. Applying Taylor formula to both cumulative normal distributions in equation (4.10) at L= ln(S/K)/(** yields

 (4.10)

 and

 (4.11)

where, e1 and e2 are the remainder terms of the Taylor’s formulas. Equations (4.10) and (4.11) are obtained by the fact that N(x) = - N′(x)x.

Given N(0) =1/2, N′(0) =1/, N‴(0) = -N′(0), N″(0) = 0 = N⁗(0). We apply the Taylor’s formula to the N[ln(S/K)/(**] and N′[ln(S/K)/(**] in equations (4.10) and (4.11) at 0

 (4.12)

 (4.13)

where e3 and e4 are the remainders of the Taylor’s formulas. Substituting (4.10)-(4.13) into (4.4), dropping all of remainder terms, equation (4.4) becomes

 (4.14)

Equation (4.14) is a quadratic equation of ** and can be rewritten as

 (4.15)

Solving ** from equation (4.15) yields

** (4.16)

Where a = 4(S+K)-(S-K)ln(S/K), b = 2(2C-S+K), c = 8ln(S/K)[(S-K)(1+(ln(S/K)/4)2) - (S+K)ln(S/K)/4] . A merit of equation (4.16) is to circumvent the ad hoc substitution present in Corrado and Miller and improve the accuracy of the ISD’s estimation.

 Since equations (4.12) and (4.13) are the Taylor series at the zero point and the remainder terms e1 to e4 in equations (4.10) to (4.11) are omitted in deriving equation (4.14), the ISD calculated by equation (4.16) is not an exact formula. Therefore, the effectiveness of using equation (4.16) to estimate the ISD depends on the deviation of the underlying asset price from the present value of exercise price. The merits of this formula are: (4.4) it is derived without arbitrary substitution (see Appendix 4A) as Corrado and Miller (1996). The Corrado and Miller’s formula, to calculate implied variance, can be defined as:]. (4.17)

All notations of Equation (4.17) are the same as Equation (4.16). The detailed explanation of this equation can be found in Appendix 4A.

Table 1 presents the excel program to calculate implied standard deviation in terms of Equation (4.16). Table 2 presents the excel program to calculate the implied standard deviation in terms of Equation (4.17).

**Table 1** Excel program to calculate implied standard deviation in terms of Equation (4.16)

|  |  |  |
| --- | --- | --- |
| **Inputs** |   |   |
| dividend (q) | 0 |   |
| stock price (S) | 161.53 |   |
| exercise price (X) | 160 |   |
| risk free rate (r) | 0.025 |   |
| time (t) | 0.0952 |   |
| **call option value (c)** | 4.111850351 |   |
| K=X\*e^(-rt) | 159.6195008 |   |
| S'=S\*e^(-qt) | 161.53 |   |
|   |   |   |
| a | 2569.150544 |   |
| b | -126.5987946 |   |
| c | 0.181850353 |   |
| delta | 14158.45106 |   |
| sigma1 | 0.154875382 | (-b+sqrt(delta))/2a\*sqrt(t) |
| sigma2 | 0.004798787 | (-b-sqrt(delta))/2a\*sqrt(t) |

|  |  |  |
| --- | --- | --- |
| **Inputs** |   |   |
| dividend (q) | 0 |   |
| stock price (S) | 161.53 |   |
| exercise price (X) | 160 |   |
| risk free rate (r) | 0.025 |   |
| time (t) | 0.095238095 |   |
| **put option value (P)** | 2.191993056 |   |
| K=X\*e^(-rt) | 159.6195008 |   |
| S'=S\*e^(-qt) | 161.53 |   |
|   |   |   |
| a | 2569.150544 |   |
| b | -126.2234794 |   |
| c | 0.181850353 |   |
| delta | 14063.56301 |   |
| sigma1 | 0.154386826 | (-b+sqrt(delta))/2a\*sqrt(t) |
| sigma2 | 0.004813973 | (-b-sqrt(delta))/2a\*sqrt(t) |

**Table 2** Excel program to calculate implied standard deviation in terms of Equation (4.17).

|  |  |
| --- | --- |
| **Estimating Implied Volatility (or ISD) in BS Option Values** |   |
|   |   |
| Share price (S) | **161.53** |
| Exercise price (X) | **160.00** |
| Int rate-cont (r) | **2.00%** |
|   |   |
| Dividend yield (q) | **0.00%** |
| Time now (0, years) | 0.00 |
| Time maturity (T, years) | **0.10** |
| Option life (T, years) | 0.10 |
| Volatility (s) | **15.49%** |
|   |   |
| iopt | **1** |
| BS Option value | 4.428610134 |
| Observed Option price | **3.37** |

|  |  |
| --- | --- |
| **ISD Estimate (Corrado & Miller)** | 11.83% |
|   |   |
| S \* Exp (-q\*T) | 161.53 |
| X \* Exp (-r\*T) | 159.70 |
| calc0 | 2.45 |
| calc1 | 4.95 |
| calc2 | 4.68 |

**Source of data for Table 1 and Table 2 comes from IBM Company from April 21st.**

**4. Formulas for Implied Standard Deviation under Different Exercise Prices**

In this section we derive two formulas to estimate the ISD from call options with different exercise price. The derived formulas can apply to a wide range of exercise prices that deviate from the underlying asset price. The advantage from using options with different exercise prices in this section is to obtain a no error’s first term in the expansion, thus, gives more accurate formulas for ISD. The first term in Taylor expansion usually provides the major component of the true function, the remaining terms provide correction to adjust and approximate the true function. For example, if the call option value C1 with exercise price E1 is expanded at the exercise price E2, the first term of the Taylor series expansion is the given option value C2 on E2. Similarly, the first term of the Taylor expansion on option value C2 on E2 at exercise price E1 is C1. Since both C1 and C2 are given, there is no need to expand the first terms again.

Consider two call options C1 and C2 on the same time to the maturity with prices of E1 and E2 (where E1 < E2 ), their present value are K1, and K2, respectively. Applying Taylor’s expansion to (4.4) at K2 for C1 and at K1 for C2 respectively yield,

**,** (4.18)

**.** (4.19)

Here ε1 and ε2 are the remainder terms of C1 at K2 and C2 at K1 from (4.4). Dividing both sides of equations (4.18) and (4.19) by (K2-K1) and simple manipulations produce the same left hand side of (C1-C2)/(K2-K1). Then applying the inverse function of cumulative normal function on both sides and after using the Taylor’s formula yields the following equations

**,** (4.20)

**,** (4.21)

where η1 and η2 are the remainder terms of Taylor’s formulas derived from (4.18) and (4.19) respectively. After dropping the remainder terms, ISD solved individually from either equation (4.20) or (4.21) could be more volatile than that solved simultaneously with both equations. Combining equations (4.20and (4.21) could cause the effects of remainder term (η1 + η2) to partially offset and thus, reduce their variability. Dropping the remainder term (η1 + η2), a simple manipulation produces the following equation

 . (4.22)

Equation (4.22) is a quadratic equation of the ISD and thus can be solved as,

 . (4.23)

There are two solutions for the ISD in equation (4.22), yet the ISD must only take one of them. It is clear that, if (C1 –C2)/(K2 –K1) >1/2, the inverse of cumulative normal function on it must be non-negative. Consequently, ISD must take the value with the positive sign in equation (4.23), or else ISD will become a negative, which violates the positive requirement for ISD. On the other hand, if (C1 –C2)/(K2 –K1) <1/2 and K1<S<K2, the ISD must take the positive sign in equation (4.23) too. The negative sign only apply when S is less than K1. It is clear that if stock price is less than the lower exercise K1 (i.e., then both call options are out-of-the-money) and if we had chosen the value with the plus sign in equation (4.23), ISD calculated by equation (4.23) will be overstated.

Furthermore, if K2 approaches K1, then (C1 - C2)/(K2 - K1) in equation (4.23) tends to -∂C/∂K and ln(S2/ K1K2) tends to zero. Given positive ISD, ISD calculated by equation (4.23) must converge to –2N-1(-∂C/∂K) (see equation (4.8)). Convergence implies that the ISD in equation (4.23) must take the negative sign for the exercise prices. Note that the ISD in equation (4.23) is not an exact solution because the remainder terms η1 and η2 were dropped out in the derivation of equation (4.22).

Equation (4.23) provides a formula to calculate the ISD. All of the parameters on the right hand sides of equation (4.23) are given. A merit of this formula is that a sufficient condition to calculate ISD by equation (4.23) only requires that there existed any two consecutive call option values with different exercise prices. Although it may not always hold that the call option value at S=K, it is easier to obtain any consecutive two traded call option values and their exercise prices in practice. In particular, equation (4.23) is applicable to a wide range of exercise prices if their call options are observable. Of course, its accuracy will depend on the magnitude of the deviation between these two exercise prices. The closer are these two exercise prices, the lower is the remainder of the Taylor Formulas in equations (4.18) and (4.19). Consequently, the ISD estimated by equation (4.23) is more accurate than in previous models. In particular, both formulas by Corrado and Miller (1996) and by Li (2005) are only valid when calculated at near at-the-money. Outside this range, both formulas’ performance is poor if not applicable.

Similar to equation (4.22), if there is a third call option C3 with the discounted exercise price K3, then the following equation (4.24) must hold for K2, K3 and C2, C3.

. (4.24)

 Given the constant variance assumption in Black and Scholes option model, the following equation (4.25) is thus derived by subtracting equations (4.23) from (4.24).

. (4.25)

An advantage of using equation (4.25) rather than equation (4.23) to estimate the ISD is to circumvent the sign issue that appears in equation (4.23). However, a drawback of using equation (4.24) is that there must exist at least three instead of two call options for equation (4.23). Equation (4.25) provide a simple formula to calculate ISD because all options values and exercises price are given and the inverse function of the standard cumulative normal function also available in the Excel spreadsheet. Instead of calculating a complex function of cosine and inverse cosine function in the formula in Li (2005), equation (4.23) provides a more convenient approach to calculate ISD./ Furthermore, the Li’s formula suffers from limited applicable range as stated before. The accuracy of the ISD calculated by equations (4.23) and (4.25) will be addressed in Section V.

1. 1 Note that N(-****) = 1 - N(****). [↑](#footnote-ref-1)
2. 2 Given S=K, the BS model becomes C=S[N(****)-N(-****)]=S[N(****)-(1-(N(****))] = -S + 2 SN(****), solve the ISD yields equation (4.7). [↑](#footnote-ref-2)
3. [↑](#endnote-ref-1)
4. [↑](#endnote-ref-2)
5. 3$ $The function used to calculate the inverse cumulative normal function is named as “NORMSINV” in *fx* of the Excel. [↑](#footnote-ref-3)
6. **5. Volatility Smile**

The existence of volatility smile is due to Black-Scholes formula cannot precisely evaluate the either call or put option value. The main reason is that the Black-Scholes formula assumes the stock price per share is log-normally distributed. If we introduce extra distribution parameters into the option pricing determination formula, we can obtain the constant elasticity volatility (CEV) option pricing formula. This formula can be found in Appendix C of this Chapter. In Appendix C we also site the empirical work done by Lee et.al (2004) to show that the CEV model performs better than the Black Scholes model in evaluating either call or put option value

 How close are the market prices of options to those predicted by the Black-Scholes-Merton model? Do traders really use the Black-Scholes-Merton model when determining a price for an option? Are the probability distributions of asset prices really lognormal? In this chapter we answer these questions. We explain that traders do use the Black-Scholes-Merton model—but not in exactly the way that Black, Scholes, and Merton originally intended. This is because they allow the volatility used to price an option to depend on its strike price and time to maturity.

 A plot of the implied volatility of an option as a function of its strike price is known as a *volatility smile.* In this chapter, we describe the volatility smiles that traders use in equity and foreign currency markets. We explain the relationship between a volatility smile and the probability distribution being assumed for the future asset price. We also discuss how option traders vary volatility with option maturity and how they use volatility surfaces as pricing tools.

 Put-call parity can be used to show that the implied volatility of a European call option must be the same as that of a European put option when both have the same strike price and time to maturity. This is convenient. It means that the volatility smile for European call options must be the same as that for European put options. The implied volatility of an American option is in most cases very similar to that of a European option with the same strike price and time to maturity. As a result, we can say that the volatility smiles we will present apply—at least approximately—to all European and American options with a particular time to maturity.

**5.1 Foreign Currency Options**

The volatility smile for foreign currency options has the general form shown in Figure 1. The implied volatility is relatively low for at-the-money options, but becomes progressively higher as the option moves either into the money or out of the money.



**Figure 1** Volatility Smile for Foreign Currency Options

 The volatility smile in Figure 1 corresponds to the probability distribution shown by the solid line if Figure 2. We will refer to this as the *implied distribution*. A lognormal distribution with the same mean and standard deviation as the implied distribution is shown by the dashed line in Figure 2. It can be seen that the implied distribution has heavier tails than the lognormal $distribution^{1}$.

 To see that Figures 1 and 2 are consistent with each other, consider first a deep out-of-the-money call option with a high strike price of $K\_{2}$. Figure 2 shows that the probability of this is higher for the implied probability distribution than for the lognormal distribution. We therefore expect the implied distribution to give a relatively high price for the option. A relatively high price leads to a relatively high implied volatility—and this is exactly what we observe in Figure 1 for the option. The two figures are therefore consistent with each other for high strike prices. Consider next a deep-out-of-the-money put option with a low strike price of $K\_{1}$. Figure 2 shows that the probability of this is also higher for the implied probability distribution than for the lognormal distribution. We therefore expect the implied distribution to give a relatively high price, and a relatively high implied volatility, for this option as well. Again, this is exactly what we observe in Figure 1.

**Figure 2** Implied distribution and lognormal distribution for foreign currency options

**Empirical Results**

We have just shown that the volatility smile used by traders for foreign currency options implies that they consider that the lognormal distribution understates the probability of extreme movements in exchange rates. To test whether they are right, Table 3 examines the daily movements in 12 different exchange rates over a 10-year $period^{2}$. The first step in the production of the table is to calculate the standard deviation of daily percentage change in each exchange rate. The next stage is to note how often the actual percentage change exceeded one standard deviation, two standard deviations, and so on. The final stage is to calculate how often this would have happened if the percentage changes had been normally distributed. (The lognormal model implies that percentage changes are almost exactly normally distributed over a one-day time period.)

 Daily changes exceed three standard deviations on 1.34% of days. The lognormal model predicts that this should happen on only 0.27% of days. Daily changes exceed four, five, and six standard deviations on 0.29%, 0.08%, and 0.03% of days, respectively. The lognormal model predicts that we should hardly ever observe this happening. The table therefore provides evidence to support the existence of heavy tails (Figure 2) and the volatility smile used by traders (Figure 1). Business Snapshot 1 shows how you could have made money if you had done the analysis in Table 3 ahead of the rest of the market.

**Table 3** Percent of days when daily exchange rate moves are greater than one, two,…,six standard deviations (S.D. = standard deviation of daily change)

|  |  |  |
| --- | --- | --- |
|  | *Real World* | *Lognormal Model* |
| $$>1 S.D.$$ | 25.04 | 31.73 |
| $$>2 S.D. $$ | 5.27 | 4.55 |
| $$>3 S.D.$$ | 1.34 | 0.27 |
| $$>4 S.D.$$ | 0.29 | 0.01 |
| $$>5 S.D.$$ | 0.08 | 0.00 |
| $$>6 S.D.$$ | 0.03 | 0.00 |

|  |
| --- |
| **Business Snapshot 1** Making money from foreign currency optionsBlack, Scholes, and Merton in their option pricing model assume that the underlying’s asset price has a lognormal distribution at future times. This is equivalent to the assumption that asset price changes over a short period of time, such as one day, are normally distributed. Suppose that most market participants are comfortable with the Black-Scholes-Merton assumptions for exchange rates. You have just done the analysis in Table 2 and know that the lognormal assumption is not a good one for exchange rates. What should you do? The answer is that you should buy deep-out-of-the-money call and put options on a variety of different currencies and wait. These options will be relatively inexpensive and more of them will close in the money than the lognormal model predicts. The present value of your payoffs will on average be much greater than the cost of the options.  In the mid-1980s, a few traders knew about the heavy tails of foreign exchange probability distributions. Everyone else thought that the lognormal assumption of Black-Scholes-Merton was reasonable. The few traders who were well informed followed the strategy we have described—and made lots of money. By the late 1980s everyone realized that foreign currency options should be priced with a volatility smile and the trading opportunity disappeared.  |

**Reasons for the Smile in Foreign Currency Options**

Why are exchange rate not lognormally distributed? Two of the conditions for an asset price to have a lognormal distribution are:

	1. The volatility of the asset is constant.
	2. The price of the asset changes smoothly with no jumps.Neither of these conditions is satisfied for an exchange rate. The volatility of an exchange rate is far from constant, and exchange rates frequently exhibit $jumps.^{3}$ The effect of both nonconstant volatility and jumps is that extreme outcomes become more likely. As we will discuss in Appendix A, the CEV model did not assume variance is constant. In other words, the variance is a function of the stock price per share. In addition, the CEV model can take care of the jump of foreign exchange better than the Black-Scholes model.

 The impact of jumps and nonconstant volatility depends on the option maturity. As the maturity of the option is increased, the percentage impact of a nonconstant volatility on prices becomes more pronounced, but its percentage impact on implied volatility usually becomes less pronounced. The percentage impact of jumps on both prices and the implied volatility becomes less pronounced as the maturity of the option is $increased.^{4}$ The result of all of this is that the volatility smile becomes less pronounced as option maturity increases.

**5.2 Equity Options**

The volatility smile for equity options has been studied by Rubinstein (1985), Rubinstein (1994) and Jackwerth and Rubinstein (1996). Prior to 1987, there was no marked volatility smile. Since 1987 the volatility smile used by traders to price equity options (both those on individual stocks and those on stock indices) has had the general form shown in Figure 3. This is sometimes referred to as *volatility skew.* The volatility decreases as the strike price increases. The volatility used to price an option with a low strike price (i.e., a deep-out-of-the-money put or deep-in-the-money call) is significantly higher than that used to price an option with a high strike price (i.e., a deep-in-the-money put or deep-out-of-the-money call).

 The volatility smile for equity options corresponds to the implied probability distribution given by the solid line in Figure 4. A lognormal distribution with the same mean and standard deviation as the implied distribution is represented by the dotted line. It can be seen that the implied distribution has a heavier left tail and less heavy right tail than the lognormal distribution. 

**Figure 3** Volatility smile for equities

|  |
| --- |
| **Business Snapshot 2** CrashophobiaIt is interesting that the pattern for equities in Figure 3 has existed only since the stock market crash of October 1987. Prior to October 1987, implied volatilities were much less dependent on strike price. This has led Mark Rubinstein to suggest that one reason for the equity volatility smile may be “crashophobia.” Traders are concerned about the possibility of another crash similar to October 1987, and they price options accordingly.  There is some empirical support for this explanation. Declines in the S&P 500 tend to be accompanied by a steepening of the skew. When the S&P 500 increases, the skew tends to become less steep.  |

 To see that Figures 3 and 4 are consistent with each other, we proceed as for Figures 1 and 2 and consider options that are deep out of the money. From Figure 4, a deep-out-of-the-money call with a strike price of $K\_{2}$ has a lower price when the implied distribution is used than when the lognormal distribution is used. This is because the option pays off only if the stock price proves to be above $K\_{2}$, and the probability of this is lower for the implied probability distribution than for the lognormal distribution. We therefore expect the implied distribution to give a relatively low price for the option. A relatively low price leads to a relatively low implied volatility—and this is exactly what we observe in Figure 3 for the option. Consider next a deep-out-of-the-money put option with a strike price of $K\_{1}$. This option pays off only if the stock price proves to be below $K\_{1}$. Figure 4 shows that the probability of this is higher for the implied probability distribution than for the lognormal distribution. We therefore expect the implied distribution to give a relatively high price, and a relatively high implied volatility, for this option. Again, this is exactly what we observe in Figure 3.



**Figure 4** Implied distribution and lognormal distribution for equity options

**The Reason for the Smile in Equity Options**

One possible explanation for the smile in equity options concerns leverage. As a company’s equity declines in value, the company’s leverage increases. This means that the equity becomes more risky and its volatility increases. As a company’s equity increases in value, leverage decreases. The equity then becomes less risky and its volatility decreases. This argument suggests that we can expect the volatility of a stock to be a decreasing function of the stock price and is consistent with Figures 3 and 4. Another explanation is “crashophobia” (see Business Snapshot 2).

**5.3 The Volatility Term Structure and Volatility Surfaces**

Traders allow implied volatility to depend on time to maturity as well as strike price. Implied volatility tends to be an increasing function of maturity when short-dated volatilities are historically low. This is because there is then an expectation that volatilities will increase.

**Table 4** Volatility Surface

|  |  |
| --- | --- |
| *Option Maturity* | *Strike Price* |
|  | 0.90 | 0.95 | 1.00 | 1.05 | 1.10 |
| 1 month | 14.2 | 13.0 | 12.0 | 13.1 | 14.5 |
| 3 months | 14.0 | 13.0 | 12.0 | 13.1 | 14.2 |
| 6 months | 14.1 | 13.3 | 12.5 | 13.4 | 14.3 |
| 1 year | 14.7 | 14.0 | 13.5 | 14.0 | 14.8 |
| 2 years | 15.0 | 14.4 | 14.0 | 14.5 | 15.1 |
| 5 years | 14.8 | 14.6 | 14.4 | 14.7 | 15.0 |

Similarly, volatility tends to be a decreasing function of maturity when short-dated volatilities are historically high. This is because there is then an expectation that volatilities will decrease,

 Volatility surfaces combine volatility smiles with the volatility term structure to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity. An example of a volatility surface that might be used for foreign currency options is shown in Table 4. (The current exchange rate is assumed to be 1.00.) In this table, the volatility smile becomes less pronounced as the option maturity increases. As mentioned earlier, this is what is observed for currency options. It is also what is observed for options on most other assets.

 One dimension of a volatility surface is strike price; the other is time to maturity. The main body of the volatility surface shows implied volatilities calculated from the Black-Scholes-Merton model. At any given time, some of the entries in the volatility surface are likely to correspond to options for which reliable market data are available. The implied volatilities for these options are calculated directly from their market prices and entered into the table. The rest of the volatility surface is typically determined using some form of interpolation.

 When a new option has to be valued, financial engineers look up the appropriate volatility in the table. For example, when valuing a nine-month option with a strike price of 1.05, a financial engineer would interpolate between 13.4 and 14.0 to obtain a volatility of 13.7%. This is the volatility that would be used in the Black-Scholes-Merton formula or in a binomial tree. When valuing a 1.5 year option with a strike price of 0.925, a two-dimensional interpolation would be used to give an implied volatility of 14.525%.

 Define *T* as the time to maturity, $S\_{0}$ as the asset price, and $F\_{0}$ as the forward price of the asset. Sometimes the volatility smile is defined as the relationship between the implied volatility and *K/*$S\_{0}$rather than between the implied volatility and *K.* A more sophisticated approach is to relate the implied volatility to

$$\frac{1}{\sqrt{T}}ln\frac{K}{F\_{0}}$$

The smile is then usually much less dependent on the time to maturity.

**The Role of the Model**

How important is an option-pricing model if traders are prepared to use a different volatility for every deal? It can be argued that the Black-Scholes-Merton model is no more than a sophisticated interpolation tool used by traders for ensuring that an option is priced consistently with the market prices of other actively traded options. If traders stopped using Black-Scholes-Merton and switched to another plausible model, the volatility surface would change and the shape of the smile would change. But arguably, the dollar prices quoted in the market would not change appreciably. Of course, the model used can affect Greek letters and are important for pricing in situations where similar derivatives do not trade.

**5.4 When a Single Large Jump is Anticipated**

We now consider an example of how an unusual volatility smile could arise in equity markets. Suppose that a stock price is currently $50 and an important news announcement in a few days is expected to either increase the stock by $8 or reduce it by $8. (This announcement might concern the outcome of a takeover attempt or the verdict in an important law suit.)

 The probability distribution of the stock price in, say, one month might then consist of a mixture of two lognormal distributions, the first corresponding to favorable news, and the second to unfavorable news. The situation is illustrated in Figure 5. The solid line shows the mixture-of-lognormals distribution for the stock price on one month; the dashed line shows lognormal distribution with the same mean and standard deviation as this distribution.



**Figure 5** Effect of a single large jump. The solid line is the true distribution; the dashed line is the lognormal distribution.



**Figure 6** Change in stock price in one month

 The true probability distribution is bimodal (certainly not lognormal). One easy way to investigate the general effect of a bimodal stock price distribution is to consider the extreme case where there are only two possible future stock prices. This is what we will now do. Suppose that the stock price is currently $50 and that it is known that in one month it will be either $42 or $58. Suppose further that the risk-free rate is 12% per annum. The situation is illustrated in Figure 6. Options can be valued using the binomial model from Chapters 12 and 18. In this case, *u* = 1.16, *d* = 0.84, *a* = 1.0101, and *p* = 0.5314. The results from valuing a range of different options are shown in Table 4. The first column shows alternative strike prices; the second shows prices of one-month European call options; the third shows prices of one-month European put options; and the fourth shows implied volatilities. (As shown in the appendix to this chapter, implied volatility of a European put option is the same as that of a European call option when they have the same strike price and maturity.) Figure 7 displays the volatility smile from Table 4. It is actually a “frown” (the opposite of that observed for currencies) with volatilities declining as we move out or into the money. The volatility implied from an option with a strike price of 50 will overprice an option with a strike price of 44 or 56.



**Figure 7** Volatility smile for situation in Table 4

The Black-Scholes-Merton model and its extensions assume that the probability distribution of the underlying asset at any given future time is lognormal. This assumption is not the one made by traders. They assume the probability distribution of an equity price has a heavier left tail and less heavy right trail than the lognormal distribution. They also assume that the probability distribution of an exchange rate has a heavier right tail and a heavier left tail than the lognormal distribution.

**Table 5** Implied volatilities in a situation where it is known that the stock price will move from $50 to either $42 or $58

|  |  |  |  |
| --- | --- | --- | --- |
| *Strike Price ($)* | *Call Price ($)* | *Put Price ($)* | *Implied Volatility (%)* |
| 42 | 8.42 | 0.00 | 0.0 |
| 44 | 7.37 | 0.93 | 58.8 |
| 46 | 6.31 | 1.86 | 66.6 |
| 48 | 5.26 | 2.78 | 69.5 |
| 50 | 4.21 | 3.71 | 69.2 |
| 52 | 3.16 | 4.64 | 66.1 |
| 54 | 2.10 | 5.57 | 60.0 |
| 56 | 1.05 | 6.50 | 49.0 |
| 58 | 0.00 | 7.42 | 0.0 |

 Traders use volatility smiles to allow for nonlognormality. The volatility smile defines the relationship between the implied volatility of an option and its strike price. For equity options, the volatility smile tends to be downward sloping. This means that out-of-the-money puts and in-the-money calls tend to have high implied volatilities whereas out-of-the-money calls and in-the-money puts tend to have low implied volatilities. For foreign currency options, the volatility smile is U-shaped. Both out-of-the-money and in-the-money options have higher implied volatilities than at-the-money options.

 Often traders also use a volatility term structure. The implied volatility of an option then depends on its life. When volatility smiles and volatility term structures are combined, they produce volatility surface. This defines implied volatility as a function of both the strike price and the time to maturity.

**6. MATLAB Approach for Calculating Implied Variance**

If we try to estimate implied standard deviation by non-linear option pricing model, we can use the MATLAB program to achieve this goal. Appendix 4B will show how the MATLAB approach can be used to estimate implied standard deviation in terms of the Black Scholes model. In addition, this approach can be used to estimate implied standard deviation in terms of the Constant Elasticity Volatility (CEV) model. The detailed discussion of the CEV model can be found in Appendix 4C.

**7. Summary**

In this Chapter we discussed Implied Variance, Volatility Smile, and the Constant Elasticity Volatility (CEV) model. These topics will be further discussed in the appendixes. In Appendix 4A, we will discuss Corrado and Miller’s Approach to Estimate Implied Standard Deviation. In Appendix 4B, we will discuss the MATLAB Approach to Estimate Implied Variance. Finally, in Appendix 4C, we will discuss Noncentral χ2 and the Option Pricing Model.

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**Appendix 4A: Corrado and Miller’s Approach to Estimate Implied Standard Deviation**

Based upon Black and Scholes’ option pricing model, Brenner and Subrahmanyam (1988) derive the following formulae of ISD under the condition that the stock price S equal to the present value K of the exercise price E:

  (4A.1)

Corrado and Miller (1996) expand the cumulative normal distribution in Black and Sholes model up to the third order and omit the rest of reminders. The resulting Taylor expansion of option pricing model on the zero point results a quadratic equation of the ISD. They then solve for the ISD from the quadratic equation and derive the following formulae (4A.2) of ISD (their equation (6) in Corrado and Miller (1996), detail see page 597).

 . (4A.2)

 Corrado and Miller stated “The accuracy of the equation (6) can be significantly improved by minimizing its concavity as follows. First, for simplicity only, we substitute the logarithmic approximation ln(S/X) ~ 2(S-X)/(S+X) into (6) and then replaced the value “4” with parameter α to obtain this restatement of the quadratic formula”.

. (4A.3)

This α is called tweaking parameter by Corrado and Miller (2004). To minimize the concavity of their equation (7) (here the (4A.3) at the stock price equal to the present value X of exercise price, they derive a formula for α in terms normal density function and cumulate normal function with ISD. They solve the α (α is a function of unknown variable of ISD, detail see their equations (8) and (9) in page 598) and then substitute α = 2 to obtain the following formula of ISD equation (4A.4) (the equation (10) in Corrado and Miller):

]. (4A.4)

As shown by Corrado and Miller, their equation (10) (here 4A.4) is more accurate than their equation (6) (here (4A.2)).

**Appendix 4B : MATLAB Approach to Estimate Implied Variance**

Usually, implied variance can be obtained from a call or put option model by an optimization technique. For each individual option, the implied variance can be obtained by first choosing an initial estimate, and then Equation (4B.1) is used to iterate towards the correct value.

 (4B.1)

Where

=market price of call option at time;

=true or actual implied standard deviation;

=initial estimate of implied standard deviation;

=theoretical price of call option  at time given ;

=partial derivative of the call option with respect to the standard deviation at ;

=error term.

The partial derivative of the call option with respective to the standard deviation from Black-Scholes model is:

 (4B.2)

It is also called Vega of option.

The iteration proceeds by reinitializing σ0 to equal σ1 at each successive stage until an acceptable tolerance level is attained. The tolerance level used is:

 (4B.3)

The MATLAB finance toolbox provides a function blsimpv to search for implied volatility. The algorithm used in the blsimpv function is Newton’s method, just as the procedure described in Equation (4B.1). This approach minimizes the difference between observed market option value and the theoretical value of B-S model, and obtain the ISD estimate until tolerance level is attained.

The complete command of the function *blsimpv* is: Volatility = *blsimpv*(Price, Strike, Rate, Time, Value, Limit, Yield, Tolerance, Class). And the command with default setting is: Volatility = *blsimpv*(Price, Strike, Rate, Time, Value).

There are nine inputs in total, while the last four of them are optional. Detailed explanations of all the inputs are as follows:

***Inputs:***

*Price - Current price of the underlying asset.*

*Strike - Strike (i.e., exercise) price of the option.*

*Rate - Annualized continuously compounded risk-free rate of return over the life of the option, expressed as a positive decimal number.*

*Time - Time to expiration of the option, expressed in years.*

*Value - Price (i.e., value) of a European option from which the implied volatility of the underlying asset is derived.*

***Optional Inputs:***

*Limit - Positive scalar representing the upper bound of the implied volatility search interval. If empty or missing, the default is 10, or 1000% per annum.*

*Yield - Annualized continuously compounded yield of the underlying asset over the life of the option, expressed as a decimal number. For example, this could represent the dividend yield and foreign risk-free interest rate for options written on stock indices and currencies, respectively. If empty or missing, the default is zero.*

*Tolerance - Positive scalar implied volatility termination tolerance. If empty or missing, the default is 1e-6.*

*Class - Option class (i.e., whether a call or put) indicating the option type from which the implied volatility is derived. This may be either a logical indicator or a cell array of characters. To specify call options, set Class = true or Class = {'Call'}; to specify put options, set Class = false or Class = {'Put'}. If empty or missing, the default is a call option.*

***Output:***

*Volatility - Implied volatility of the underlying asset derived from European option prices, expressed as a decimal number. If no solution can be found, a NaN (i.e., Not-a-Number) is returned.*

**Example:**

Consider a European call option trading at $5 with an exercise price of $95 and 3 months until expiration. Assume the underlying stock pays 5% annual dividends, which is trading at $90 at this moment, and the risk-free rate is 3% per annum. Under these conditions, the command used in Matlab will be either of the following two:

Volatility = blsimpv(90, 95, 0.03, 0.25, 5,[],0.05,[], { 'Call'})

Volatility = blsimpv(90, 95, 0.03, 0.25, 5,[],0.05,[], true)

Note that this function provided by MATLAB’s toolbox can only estimate implied volatility from a single option. For more than one option, the user needs to write their own programs to estimate implied variances.

**Appendix 4C: Noncentral χ2 and the Option Pricing Model**

It is well known that  is distributed as χ2 with n degree of freedom. This is a central χ2 distribution. It can be shown that  is distributed as noncentral χ2 with n degree of freedom and a noncentral parameter



If *μ* = 0, the distribution of  reduces to the central χ2distribution.

Black-Scholes’ Option Pricing Model assumed that the variance of stock rate of return (*σ*2) is constant. If the variance of stock rate of return is a function of stock price per share, *σ*2*Sβ*–2,then the Option Pricing Model can be generalized as

 (4C.1)

Where is a cumulative non-central Chi-Square density function

That is C = S\*(1-cumulative non-central Chi-Square density function) – \*cumulative non-central Chi-Square density function.

 (4C.2)

where ***T*** *=* time of expiration of option, ***t***= current time, ***r*** = risk-free rate. χ2(*W*, *V*, λ) is the cumulative non-central chi-square distribution function with *W, V,* and, λ being the upper limit of the integral, degree of freedom and noncentrality respectively. In addition *m*, *n* and K can be defined as:





  (4C.3)

Now, we discuss three possible special cases associated with Equations 4C.1 and 4C.2.

	1. If *β* ***=*** 2*,* both ***m*** and ***n***approach infinity. Then it can be shown that both Equations 4C.1 and 4C.2 reduce to the Black-Scholes Option pricing formula. That is cumulative density function of non-central Chi-Square distribution reduces to cumulative density function of normal distribution.
	2. If *β =* 1, it can be shown that Equations 4C.1 and 4C.2 reduce to

where





*N*(*y*1)and *N*(*y*2)= cumulative standardized normal distribution function in terms of *y*1and *y*2respectively.

*n*(*y*1)and *n*(*y*2)= standardized normal density function in terms of *y*1, and *y*2respectively.

	1. If *β =* 0, it can be shown that Equations 4C.1 and 4C.2 reduce to  (4C.5)

where







The elasticity of variance (*σ*2*Sβ*–2) with respect to stock price per share *S* is

 (4C.6)

This implies that the Option Pricing Model defined in Equations 4C.1 and 4C.2 is a constant elasticity of variance (CEV) type of OPM.

The CEV type of option pricing model can be reduced to the following special models.

	1. *β* = 2, Equations 4C.1 and 4C.2 reduce to the Black-Scholes model
	2. *β =* 1, Equations 4C.1 and 4C.2 reduce to the absolute model as defined in Equation 4C.4
	3. *β* = 0, Equations 4C.1 and 4AC.2 reduce to the square root model as defined in Equation 4C.5**Comparing the performance of Black Scholes model and the CEV model**

Lee et.al (2004) applies an efficient algorithm for computing non-central chi-square distribution function to the CEV option pricing model. The Black-Scholes model prices are used as a benchmark for comparison. Three traditional measures are used to compare the sample fit and forecasting accuracy of the CEV model and the Black-Scholes model. First, the root mean squared error (RMSE) between the market and the model fitted (in sample) or model predicted (out of sample) option prices:

$RMSE= \sqrt{\frac{1}{n }} \sum\_{i=1}^{n}(C\_{market }- C\_{model })^{2}$. (4C.7)

Second, the mean absolute error (MAE) between the market and the model fitted (in sample) or model predicted (out of sample) option prices:

$MAE= \frac{1}{n} \sqrt{\sum\_{i=1}^{n}\left|C\_{Market}- C\_{Model}\right|}.$ (4C.8)

Third, the mean absolute percentage error (MAPE) between the market and the model fitted (in sample) or model predicted (out of sample) option prices:

$MAPE= \frac{1}{n} \sqrt{\sum\_{i=1}^{n}\left|\frac{C\_{Market}- C\_{Model}}{C\_{market}}\right|}.$ (4C.9)

When viewing the results it should be remembered that when the differences between the market prices and model prices are measured in absolute value using the mean absolute error (MAE) they are larger for in-the-money options because the option prices are usually greater for in-the-money options. When the differences are measured as a percentage of market prices using the mean absolute percentage error (MAPE), they are larger for out-of-the-money options because the market prices are usually small for out-of-the-money options.

Three benchmark measures are used to compare the performance of the Black-Scholes model and the CEV model. They are root mean squared errors (RMSE), mean absolute percentage errors (MAPE), and mean absolute error (MAE). The CEV model outperforms the BS model in all three measures. Table C1 lists the in-sample moneyness-based RMSE, MAE, MAPE values for the Black-Scholes model and the CEV model. The CEV model outperforms the BS model in the out-of-the-money category particularly. Table C2 lists the in-sample moneyness and maturity based RMSE, MAE, MAPE values for the Black-Scholes model and the CEV model. The CEV model outperforms BS model in most categories, especially the out-of-the-money category.

**Table C1.** In-sample moneyness-based RMSE, MAE, MAPE values for the Black-Scholes model and the CEV model.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Moneyness (S/K) | Model | RMSE | MAE | MAPE |
| $$DITM (\geq 1.06)$$ | BSCEV | 1.75561.7704 | 1.18891.2044 | 0.02070.0209 |
| $ITM (1.03$-1.06) | BSCEV | 1.28721.2203 | 0.87120.8311 | 0.02920.0279 |
| $$ATMI \left(1.00-1.03\right)$$ | BSCEV | 0.92630.8028 | 0.63640.5733 | 0.04090.0367 |
| $$ATMO (0.97-1.00)$$ | BSCEV | 0.81550.5655 | 0.57750.4216 | 0.11530.0855 |
| $$OTM (0.94-0.97)$$ | BSCEV | 0.88340.5112 | 0.61590.3692 | 0.22600.1706 |
| $$DOTM (<0.94)$$ | BSCEV | 1.35750.7098 | 0.89710.4877 | 0.32300.1837 |

**Table C2.** In-sample moneyness and maturity based RMSE, MAE, MAPE values for the Black-Scholes model and the CEV model.

 [↑](#endnote-ref-3)