Implementation Problems and Solutions in Stochastic Volatility Models of the Heston Type

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August 2007

ABSTRACT

In Heston's stochastic volatility framework, the main problem for implementing Heston's semi-analytic formulae for European-style financial claims is the inverse Fourier integration. The numerical integration scheme of a logarithm function with complex arguments has puzzled practitioners for many years. Without good implementation procedures, the numerical results obtained from Heston's formulae may not be robust, even for customarily-used Heston parameters, as the time to maturity is increased. In this paper, we compare three major approaches to solve the numerical instability problem inherent in the fundamental solution of the Heston model.

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I. Introduction

The randomness of the variance process varies as the square root of variance in the Heston's stochastic volatility framework. The literature on asset pricing using the Heston model has expanded dramatically over the last decade to successfully describe the empirical leptokurtic distributions of asset price returns. However, the implementations of Heston's formulae are not as straightforward as they may appear and most numerical procedures are not reported in detail (see Lee [2005]).

The complex logarithm contained in the formula of the Heston model is the primary problem. This paper compares three main approaches to this problem: rotation-corrected angle, direct integration, and simple adjusted formula. Recently, the robustness of Heston's formula has become one of the main issues on option pricing. It is a well-known fact that the logarithm of a complex variable $z = re^{i\theta}$ is multi-valued, i.e., $\ln z = \ln|z| + i(\arg(z) + 2\pi n)$ where $\arg(z) \in [-\pi, \pi)$ and $n \in \mathrm{Z}$. If one restricts the logarithm to its principal branch by setting $n = 0$ (similar to most software packages, such as C++, Gauss, Mathematica, and others), it is necessarily discontinuous at the cut (see Figure 1).

The Heston model is represented in Lewis' illustration, in which the type of financial claim is entirely decoupled from the calculation of the Green function. Different payoffs are then managed through elementary contour integration over functions and contours that depend on the payoff. In this way, one can see that the issue is fundamentally related to the Green function component of the solution. Once the implementation problems in the Green function component of the solution have been solved, the robustness of the formulae for all European-style financial claims in Heston's model can be assured.

The rest of this article proceeds as follows: Section II gives the derivation of the transformed-based solution for Heston's stochastic volatility model and introduces the discontinuity problem arising from the derived formula. Section III compares three main solutions to the discontinuity problem and gives some numerical examples to illustrate their usefulness. Section IV concludes the paper.

II The Transform-based Solution for Heston's Stochastic Volatility Model

Heston's stochastic volatility model is based on the system of stochastic differential equations, which represent the dynamics of the stock price and variance processes under the risk-neutral measure

$$dS_t = rS_t dt + S_t \sqrt{V_t}\, dW_S(t) \tag{1}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t}\, dW_V(t)\,. \tag{2}$$

S_t and V_t denote the stock price and its variance at time t, respectively; r is the risk-free interest rate. The variance evolves according to a square-root process: θ is the long-run mean variance, κ is the speed of mean reversion, and σ_V is the parameter which controls the volatility of the variance process. W_S and W_V are two standard processes of Brownian motion having the correlation ρ. The Heston partial differential equation for a European-style claim $C(S_t, V_t, t)$ with expiration T, is

$$\frac{\partial C}{\partial t}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\sigma_V\rho SV\frac{\partial^2 C}{\partial S\partial V}+\frac{1}{2}\sigma_V^2 V\frac{\partial^2 C}{\partial V^2}+rS\frac{\partial C}{\partial S}+\kappa(\theta-V)\frac{\partial C}{\partial V}-rC=0. \quad (3)$$

The fundamental transform method proposed by Lewis [2000] can reduce (3) from two variables to one and entirely separate every (volatility independent) payoff function from the calculation of the Green function. After substituting the following, $\tau=T-t$, $x=\log(S)+r(T-t)$, $C(S,V,t)=W(x,V,\tau)e^{-r(T-t)}$ into (3), we have

$$\frac{\partial W}{\partial \tau}=\frac{1}{2}V\left(\frac{\partial^2 W}{\partial x^2}-\frac{\partial W}{\partial x}\right)+\rho\sigma_V V\frac{\partial^2 W}{\partial x\partial V}+\frac{1}{2}\sigma_V^2 V\frac{\partial^2 W}{\partial V^2}+\kappa(\theta-V)\frac{\partial W}{\partial V}. \quad (4)$$

Let $G(\phi,V,\tau)$ denote the Fourier transform of $W(x,V,\tau)$:

$$G(\phi,V,\tau)\equiv\int_{-\infty}^{\infty}e^{i\phi x}W(x,V,\tau)dx. \quad (5)$$

Given the transform $G(\phi,V,\tau)$, the inversion formula is

$$W(x,V,\tau)=\frac{1}{2\pi}\int_{i\,\mathrm{Im}[\phi]-\infty}^{i\,\mathrm{Im}[\phi]+\infty}e^{-i\phi x}G(\phi,V,\tau)d\phi \quad (6)$$

so that differentiation w.r.t. x becomes multiplication by $-i\phi$ in the transform. By taking the τ derivative of both sides of (5), and then replacing $\partial W/\partial\tau$ inside the integral by the right-hand side of (4), we translate (4) into a PDE for $G(\phi,V,\tau)$.

$$\frac{\partial G}{\partial \tau}=\frac{1}{2}\sigma_V^2 V\frac{\partial^2 G}{\partial V^2}-\frac{1}{2}V(\phi^2-i\phi)G+\left(\kappa(\theta-V)-i\phi\sigma_V\rho V\right)\frac{\partial G}{\partial V} \quad (7)$$

Hence, a solution $G(\phi,V,\tau)$ to (7), which satisfies $G(\phi,V,0)=1$, is called a fundamental transform. Given the fundamental transform, a solution for a particular payoff can be obtained by

$$C(S_t, V_t, t) = \frac{1}{2\pi} e^{-r(T-t)} \int_{i\,\mathrm{Im}[\phi]-\infty}^{i\,\mathrm{Im}[\phi]+\infty} e^{-i\phi x} \tilde{W}(\phi, V, 0) G(\phi, V, \tau) d\phi \,, \tag{8}$$

where $\tilde{W}(\phi, V, 0)$ is the Fourier transform of the payoff function at maturity.

We will deal with a few common types of payoff functions and see what restrictions are necessary for their Fourier transforms to exist.

■Call Option

At maturity, the payoff of a vanilla call option with strike K is $Max[S_T - K, 0]$ in terms of our original variables. In terms of the logarithmic variables, we have $W(x, V, 0) = Max[e^x - K, 0]$, so the Fourier transform of the payoff is of the form:

$$\tilde{W}(\phi, V, 0) = \int_{-\infty}^{\infty} e^{i\phi x} W(x, V, 0) dx = \int_{\log[K]}^{\infty} e^{i\phi x} \left(e^x - K\right) dx = K^{1+i\phi} \Big/ \left(i\phi - \phi^2\right), \tag{9}$$

which does not exist unless $\mathrm{Im}[\phi] > 1$.

■Put Option

The payoff of a vanilla put option with strike K is $Min[K - S_T, 0]$. Its transformed payoff is also $K^{1+i\phi} / \left(i\phi - \phi^2\right)$, but the restriction is $\mathrm{Im}[\phi] < 0$.

■Digital Call

The payoff of a digital call with strike K is $H[S_T - K]$ where H is a Heaviside function. Its transformed payoff is $-K^{i\phi}/(i\phi)$, subject to $\mathrm{Im}[\phi] > 0$.

■Cash-secured Put

The payoff of a cash-secured put with strike K is $Min[S_T, K]$. Its transformed payoff is $K^{1+i\phi}/(\phi^2 - i\phi)$, subject to $0 < \mathrm{Im}[\phi] < 1$.

The fundamental solution of (7) is in the form

$$G(\phi, V, \tau) = e^{A(\tau,\phi)+B(\tau,\phi)V}. \tag{10}$$

After substituting (10) into (7), a pair of ordinary differential equations for $A(\tau,\phi)$ and $B(\tau,\phi)$ is obtained

$$\dot{A} = \theta\kappa B \tag{11}$$

$$\dot{B} = \frac{1}{2}B^2\sigma_V^2 - B(\kappa + i\phi\sigma_V\rho) - \frac{1}{2}(\phi^2 - i\phi). \tag{12}$$

The solutions can be expressed by

$$B(\tau,\phi) = \frac{(\kappa + i\rho\sigma_V\phi + d(\phi))}{\sigma_V^2}\frac{(1 - e^{d(\phi)\tau})}{(1 - g(\phi)e^{d(\phi)\tau})} \tag{13}$$

$$A(\tau,\phi) = \frac{\kappa\theta}{\sigma_V^2}\left((\kappa + i\rho\sigma_V\phi + d(\phi))\tau - 2\log\left[\frac{g(\phi)e^{d(\phi)\tau} - 1}{g(\phi) - 1}\right]\right) \tag{14}$$

using the auxiliary functions

$$g(\phi) = \frac{\kappa + i\rho\sigma_V\phi + d(\phi)}{\kappa + i\rho\sigma_V\phi - d(\phi)},\quad d(\phi) = \sqrt{(\phi^2 - i\phi)\sigma_V^2 + (\kappa + i\rho\sigma_V\phi)^2}. \tag{15}$$

If the complex, multi-valued logarithm is restricted to the principal branch only, discontinuities are necessarily incurred at the cut of the complex logarithm along the integration path, resulting in an incorrect value for Heston's formula. Figure 2 illustrates the discontinuity problem in the implementation of the fundamental solution. In this example, depicted in Figure 2, $S_0 = 100$, $r = 0.0319$, $V_0 = 0.010201$, $\rho = -0.70$, $\kappa = 6.21$, $\theta = 0.019$, $\sigma_V = 0.61$, and $\mathrm{Im}[\phi] = 2$.

Reasonable parameters in practice may incur the numerically-induced discontinuity such that the correct treatment of the phase jump is very crucial. In fact, in examples with long maturity periods, discontinuities are certain to arise from the formula presented in equation (14) for $A(\tau,\phi)$, if the complex logarithm uses the principal branch only and $\kappa\theta/\sigma_V^2$ is not an integer (see Figure 3).

One may shift the problem from the complex logarithm to the evaluation of

$$G(\phi,V,\tau) = \left(\frac{g(\phi)e^{d(\phi)\tau} - 1}{g(\phi) - 1}\right)^{-2\alpha} e^{\alpha(\kappa + i\rho\sigma_V\phi + d(\phi))\tau + B(\tau,\phi)V}, \tag{16}$$

where $\alpha = \kappa\theta/\sigma_V^2$. However, this formula comes with the related branch switching problem of the complex power function and discontinuities do not diminish in its implementation. Note that taking a complex variable z to the power α gives

$$z^\alpha = r^\alpha e^{i\alpha\theta}. \tag{17}$$

After restricting $\arg(z) \in [-\pi, \pi)$, the complex plane is cut along the negative real axis. Whenever z crosses the negative real axis, the sign of its phase changes from $-\pi$ to π. Therefore, the phase of z^α changes from $-\alpha\pi$ to $\alpha\pi$. This may lead to a jump because

$$e^{i\pi} = e^{-i\pi} \Rightarrow \begin{cases} e^{i\alpha\pi} \neq e^{-i\alpha\pi} & if\ \alpha \notin \mathrm{Z} \\ e^{i\alpha\pi} = e^{-i\alpha\pi} & if\ \alpha \in \mathrm{Z} \end{cases}. \tag{18}$$

To demonstrate this, Figure 4 gives the scenario that $\alpha \in \mathrm{Z}$ and there is no jump at all. Figure 5 gives another scenario that $\alpha \notin Z$ and the complex power function indeed incurs jumps.

III Solutions to the Discontinuity Problem of Heston's Formula

In the literature, various authors propose the idea of carefully keeping track of the branch by monitoring the complex logarithm function for each step along a discretised integral path to remedy phase jumps. As described in Kruse and Nögel [2005], if the imaginary value of the complex logarithm for one step differs from the previous one by more than 2π, the jump of 2π is added or subtracted to recover the continuity of phase. However, using this approach, the already complex integrals of Heston's formula may become too complicated in practice. Therefore, simulation is also considered as a practical alternative for finding option prices (see Broadie and Kaya [2004]).

To make matters worse, discontinuities arise quite naturally for customarily-used Heston parameters simply as time to maturity is increased, thereby illustrating the importance of the correct treatment of phase jumps for Heston's formula. Kahl and Jäckel [2005] remedied these discontinuities using the rotation-corrected angle of the phase of a complex variable. Shaw [2006] dealt with this problem by replacing the call to the complex logarithm by direct integration of the differential equation. In addition, Guo and Hung [2007] also proposed a simple adjusted formula to solve this discontinuity problem. From a computational and convenience point of view, the solution of Guo and Hung [2007] can be implemented easily and is thereby suitable for practical application. These solutions are presented by the following statements.

■Rotation-Corrected Angle

In order to guarantee the continuity of $A(\tau,\phi)$, an rotation-corrected term must be additionally calculated in advance. First, we introduce the notation

$$g(\phi) = r_g(\phi)e^{i\theta_g(\phi)}, \tag{19}$$

$$d(\phi) = a_d(\phi) + ib_d(\phi). \tag{20}$$

The next step is to have a closer look at the denominator of $\left(g(\phi)e^{d(\phi)\tau} - 1\right)/\left(g(\phi)-1\right)$:

$$g(\phi) - 1 = r_g(\phi)e^{i\theta_g(\phi)} - 1 = r_g^*(\phi)e^{i(\chi_g^*(\phi)+2\pi m)} \tag{21}$$

where

$$m = \operatorname{int}\left[\frac{r_g(\phi)+\pi}{2\pi}\right], \tag{22}$$

$$\chi_g^*(\phi) = \arg(g(\phi)-1), \tag{23}$$

$$r_g^*(\phi) = |\, g(\phi) - 1\,|, \tag{24}$$

and with $\mathrm{int}[\bullet]$ denoting Gauss's integer brackets. The same calculation is done with the numerator:

$$g(\phi)e^{d(\phi)\tau}-1$$

$$=r_g(\phi)e^{i\theta_g(\phi)}e^{(a_d(\phi)+ib_d(\phi))\tau}-1$$

$$=r_g(\phi)e^{a_d(\phi)\tau}e^{i(\theta_g(\phi)+b_d(\phi)\tau)}-1$$

$$=r_{gd}^*(\phi)e^{i(\chi_{gd}^*(\phi)+2\pi n)} \tag{25}$$

where

$$n=\mathrm{int}\left[\frac{\theta_g(\phi)+b_d(\phi)\tau+\pi}{2\pi}\right], \tag{26}$$

$$\chi_{gd}^*(\phi)=\arg\left(g(\phi)e^{d(\phi)\tau}-1\right), \tag{27}$$

$$r_{gd}^*(\phi)=|g(\phi)e^{d(\phi)\tau}-1|. \tag{28}$$

Hence, we can compute the logarithm of $\left(g(\phi)e^{d(\phi)\tau}-1\right)/\left(g(\phi)-1\right)$ quite simply as

$$\log\left[\frac{g(\phi)e^{d(\phi)\tau}-1}{g(\phi)-1}\right]=\log\left[r_{gd}^*(\phi)/r_g^*(\phi)\right]+i\left[\chi_{gd}^*(\phi)-\chi_g^*(\phi)+2\pi(n-m)\right], \tag{29}$$

where $2\pi(n-m)$ is the rotation-corrected angle.

■Direct Integration

Another way to avoid the branch cut difficulties arising from the choice of the branch of the complex logarithm is to perform direct numerical integration of $A(\tau,\phi)$ w.r.t. τ according to (11). Given $B(\tau,\phi)$, $A(\tau,\phi)$ can be obtained by

$$A(\tau,\phi)=\theta\kappa\int_0^\tau B(s,\phi)ds\,. \tag{30}$$

After replacing the call to the complex logarithm by direct integration of the differential equation, the complex logarithm can not be a problem any more and the continuity of $A(\tau,\phi)$ is guaranteed.

■Simple Adjusted Formula

Here, we briefly introduce the simple adjusted formula of Guo and Hung [2007] to the discontinuity problem in the implementation of Heston's formula. The solution is to move $\exp[-d(\phi)\tau]$ into the logarithm of $A(\tau,\phi)$ by simply adjusting $A(\tau,\phi)$ as follows:

$$A(\tau,\phi)=\frac{\kappa\theta}{\sigma_V^2}\left(\left(\kappa+i\rho\sigma_V\phi-d(\phi)\right)\tau-2\log\left[\frac{g(\phi)e^{d(\phi)\tau}-1}{g(\phi)-1}e^{-d(\phi)\tau}\right]\right). \tag{31}$$

The insight in Formula (31) is that the subtraction of the number 1 from a complex variable, c, results simply in a shift parallel to the real axis. Because an imaginary component must be added to move a complex number across the negative real axis, the phases of $c-1$ and c exist on the same phase interval. Therefore, the logarithms of $c-1$ and c have the same rotation count number. It illuminates this simple solution to assure that the phase of $A(\tau,\phi)$ is continuous, without the necessity of a rotation-corrected term.

The logarithm presented in (31) is the only term possibly giving rise to discontinuity. Trivially,

$$\log\left[\frac{g(\phi)e^{d(\phi)\tau}-1}{g(\phi)-1}e^{-d(\phi)\tau}\right]=\log\left[g(\phi)e^{d(\phi)\tau}-1\right]-\log\left[\left(g(\phi)-1\right)e^{d(\phi)\tau}\right]. \tag{32}$$

Because the subtraction of 1 does not affect the rotation count of the phase of a complex variable, $\log\left[g(\phi)e^{d(\phi)\tau}-1\right]$ has the same rotation count number as $\log\left[g(\phi)e^{d(\phi)\tau}\right]$. Nevertheless, $\log\left[g(\phi)e^{d(\phi)\tau}\right]-\log\left[\left(g(\phi)-1\right)e^{d(\phi)\tau}\right]$ needs no rotation-corrected terms for all levels of Heston parameters because

$$\log\left[\frac{g(\phi)e^{d(\phi)\tau}}{\left(g(\phi)-1\right)e^{d(\phi)\tau}}\right]=\log\left[\frac{g(\phi)}{g(\phi)-1}\right]=\log\left[g(\phi)\right]-\log\left[g(\phi)-1\right] \tag{33}$$

and, again, $\log\left[g(\phi)\right]$ and $\log\left[g(\phi)-1\right]$ has the same rotation count number. Hence, the formula in (31), for $A(\tau,\phi)$, provides a simple solution to the discontinuity problem for Heston's stochastic volatility model.

Compared to the rotation-corrected angle method, the simple adjusted-formula method needs no rotation-corrected terms in the already complex integral of Heston's formula to recover its continuity for all levels of Heston parameters. Although the direct integration method neither needs the rotation-corrected terms to guarantee the continuity of Heston's formula, it inevitably introduces the discretization bias into the evaluation of the Green function component of the solution. This bias may create another serious problem of computation. Many steps may be necessary to reduce the bias to an acceptable level and, hence, more computational effort is needed to guarantee that the bias is small enough. As a consequence, the direct integration method requires more computing time than the simple adjusted-formula method to avoid the discontinuity problem arising from the complex logarithm.

Figure 6 illustrates a comparison of the computing time for applying the simple adjusted-formula and direct integration methods to evaluate a European call option. The direct integration method is more time-consuming than the simple adjusted-formula method. Moreover, the simple adjusted-formula method has an advantage in that its time consumption remains almost at the same level as the time to maturity increases. In contrast, the computing time via the direct integration method increases rapidly with an increase in time to maturity. These computational results were performed on a desktop PC with an Intel Pentium D 3.4 GHz processor and 1 GB of RAM, running Windows XP Professional. The codes were written using the Mathematica software.

Table 1 is an illustration of the usefulness of the simple adjusted formula for evaluating European call options in Heston's model using the complex logarithm restricted to the principal branch. The algorithm was verified using Monte Carlo simulation with the exact method proposed by Broadie and Kaya [2004] for the stochastic volatility process. Although the exact method of simulation has the advantage that its convergence rate is much faster than that of the conventional Euler discretization method, it is, of course, computationally more burdensome than the simple adjusted-formula method.

IV. Conclusions

This paper looks at the issue raised by branch cuts in the transform solutions for European-style financial claims in the Heston model. The multi-valued nature of the complex logarithm and power functions results in numerical instability in the implementation of the fundamental transform. Compared to the work of Kahl and Jäckel [2005], neither the direct integration method of Shaw [2006] nor the simple adjusted formula of Guo and Hung [2007] requires rotation-corrected terms to assure the robustness of the evaluation of Heston's formulae. After taking computing time into consideration, the evidence shows that the simple adjusted-formula method is greatly superior to the direct integration method.

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Figures

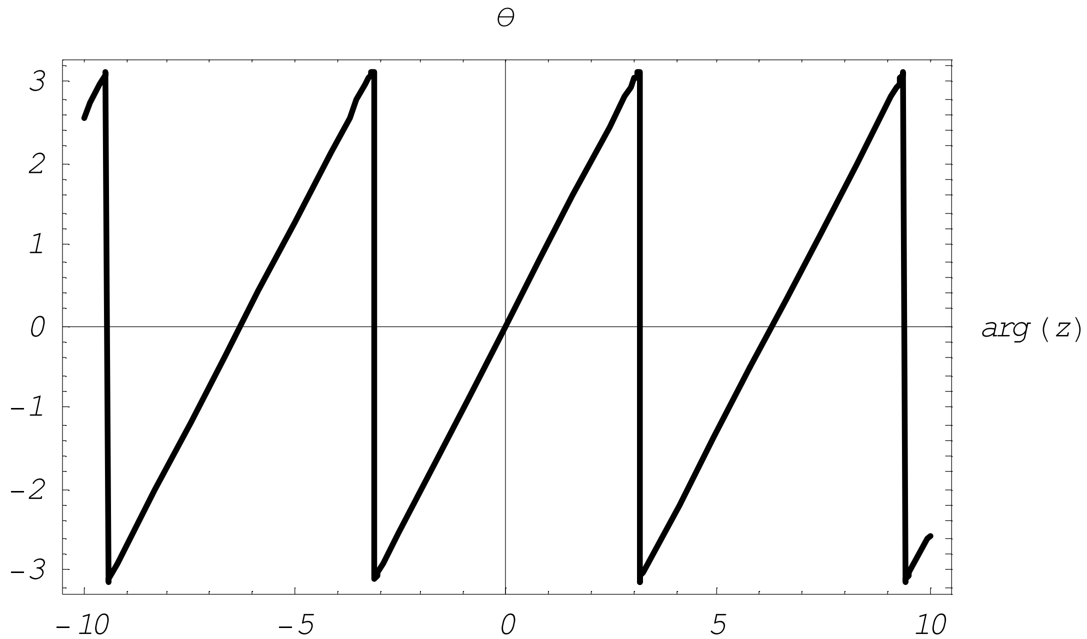

Figure 1. Discontinuities occur at the cut by restricting the logarithm of a complex variable $z = e^{i\theta}$ to the principal branch.

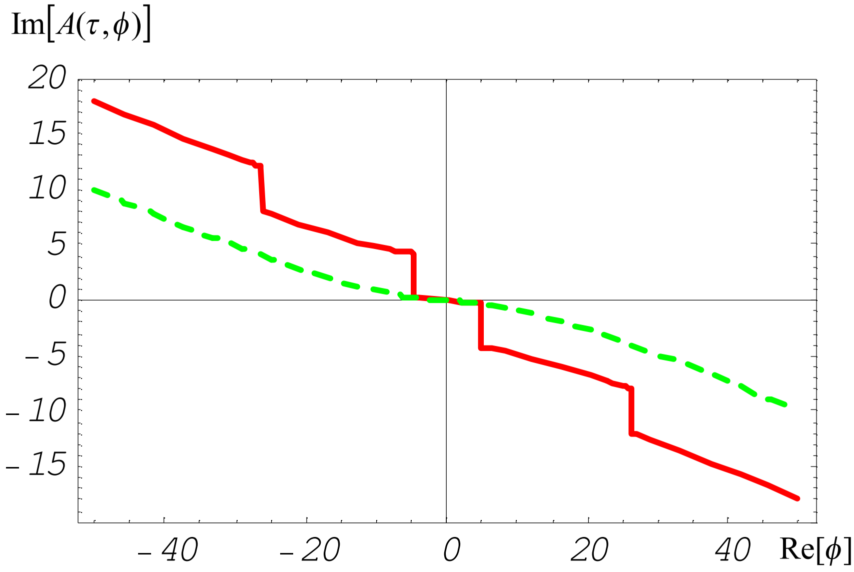

Figure 2. The discontinuity occurs in the fundamental solution of the Heston model if the logarithm with complex arguments is restricted to the principal branch. Underlying: $dS_t = rS_t dt + \sqrt{V_t} S_t dW_S(t)$ with $S_0 = 100$ and $r = 0.0319$. Variance: $dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_V(t)$ with $V_0 = 0.010201$, $\kappa = 6.21$, $\theta = 0.019$, $\sigma_V = 0.61$, and $\rho = -0.70$. Time to maturity: $T = 2.00$. The red line was obtained by evaluating $A(\tau,\phi)$ with the unfixed form given in (14). The green dashed line was obtained by evaluating $A(\tau,\phi)$ with the adjusted formula given in (31), and is the correct curve. The logarithmic function for both cases is restricted to using only the principal branch.

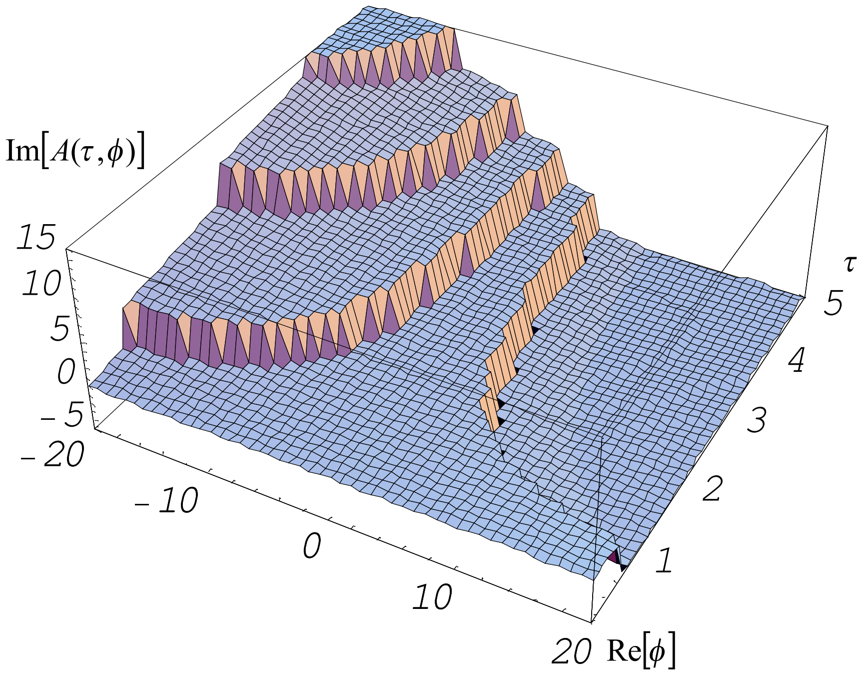

Figure 3. Discontinuities arise quite naturally for customarily-used Heston parameters, typically occurring in practice as time to maturity is increased. The other parameters are the same as those specified in Figure 2.

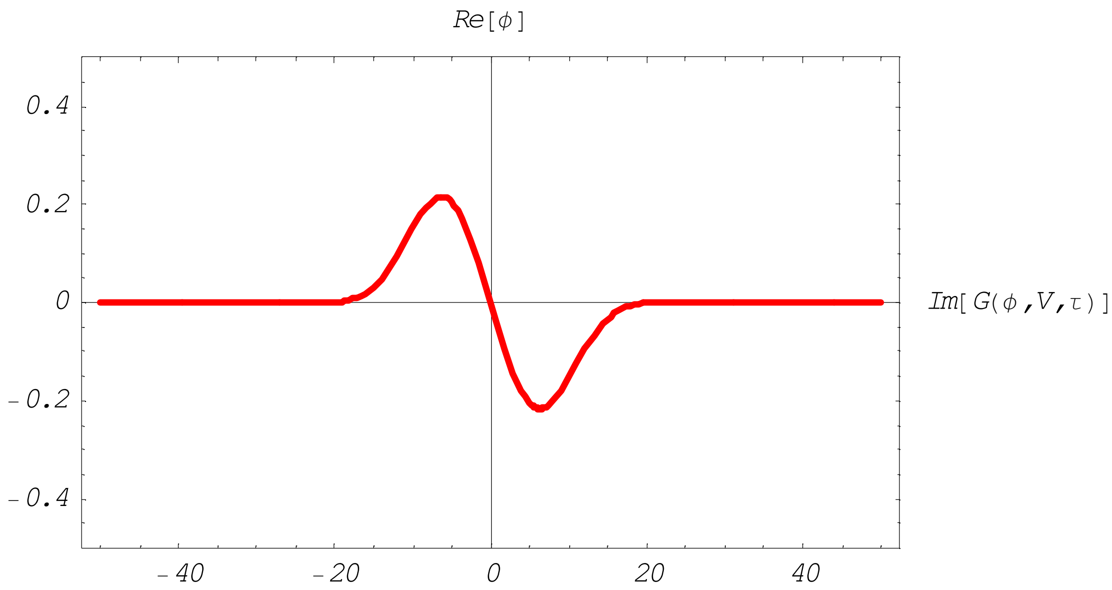

Figure 4. If the power of a complex variable z^{α} **is restricted to the principal branch,** $\alpha \in Z$ **makes discontinuities diminish at the cut**. The fundamental function $G(\phi, V, \tau)$ in (16) is evaluated with the same parameters specified in Figure 2 except for $\kappa = 19.58421$. In this scenario, $\alpha = \kappa\theta / \sigma_V^2 = 1$.

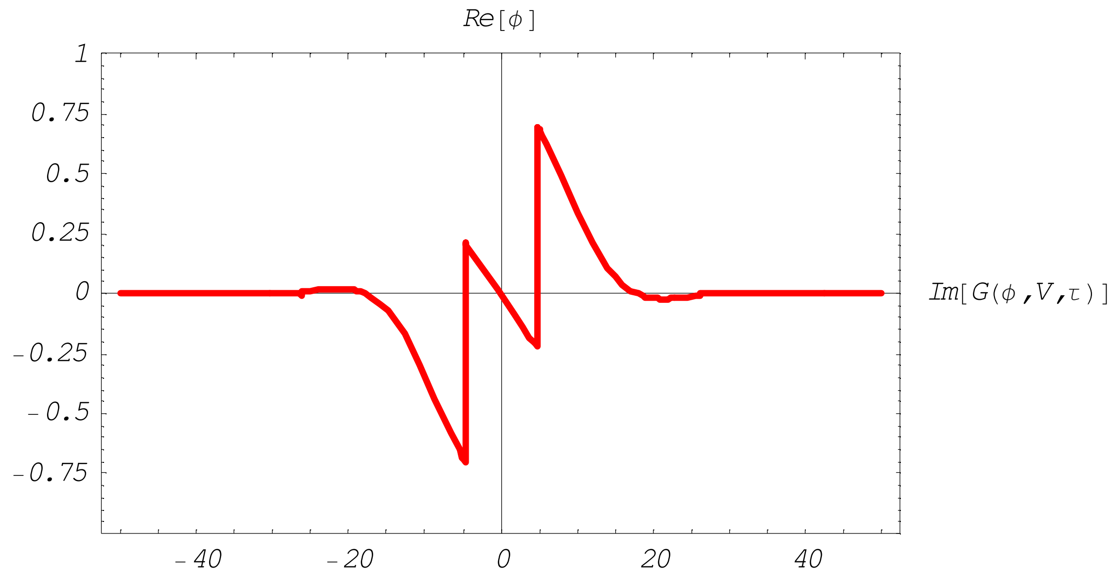

Figure 5. If the power of a complex variable z^{α} is restricted to the principal branch, $\alpha \notin Z$ makes discontinuities occur at the cut. The fundamental function $G(\phi, V, \tau)$ in (16) is evaluated with the same parameters specified in Figure 2. In this scenario, $\alpha = \kappa\theta / \sigma_V^2 = 0.3170921$.

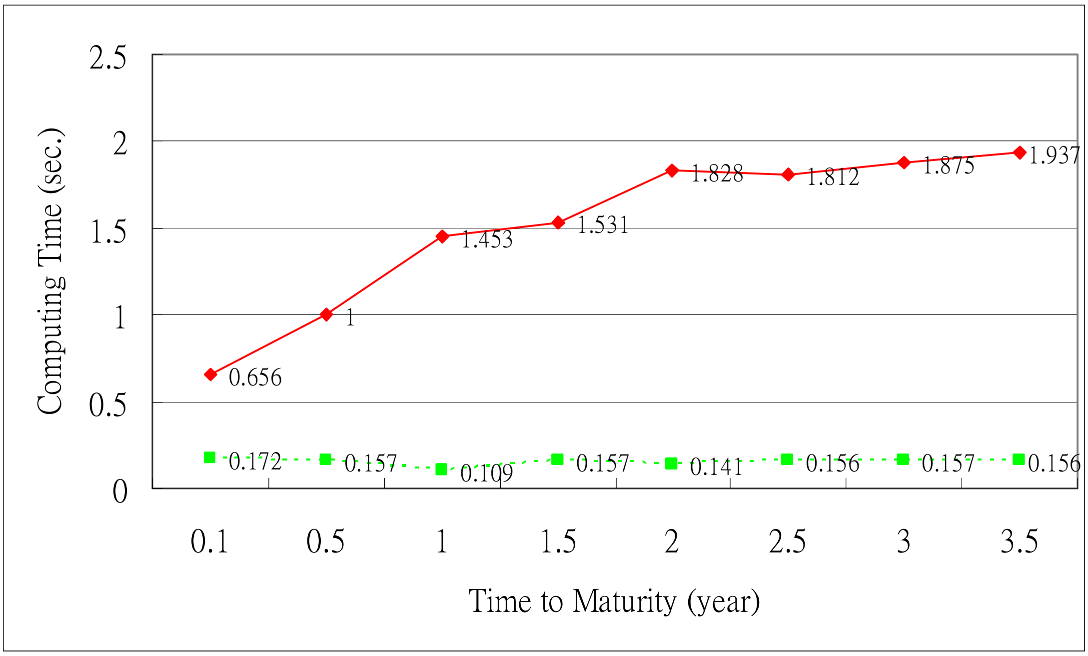

Figure 6. Computing time comparison under Heston's stochastic volatility model for a European call option: direct integration versus simple adjusted formula. The red line represents the computing time for evaluating a standard call option price using $A(\tau,\phi)$ via direct integration w.r.t. τ given in (30). The green dashed line represents the computing time for evaluating the same call option price using $A(\tau,\phi)$ via the simple adjusted formula given in (31). The other parameters are the same as those specified in Figure 2.

Table 1. Impact of the discontinuity problem on the evaluation of European call options in Heston's model on stochastic volatility.

T (year)	Monte Carlo Simulation with the Exact Method (10000 trials)	Fundamental Solution of the Heston Model	
		Adjusted Formula Using Formula (31)	Unfixed Formula Using Formula (14)
0.50	4.2658	4.2545	4.2555
1.00	6.7261	6.8061	6.4483
1.50	8.9510	8.9557	8.3286
2.00	10.9633	10.8830	9.7079
2.50	12.6100	12.6635	10.5542
3.00	14.2591	14.3366	10.9778

Here: $S_0 = 100.00$, $K = 100.00$, $r = 0.0319$, $V_0 = 0.010201$, $\rho = -0.70$, $\kappa = 6.21$, $\theta = 0.019$, and $\sigma_V = 0.61$. Note that the evaluation of the fundamental solution of the Heston model using formula (31) still yields values consistent with those of the Monte Carlo simulation for all time-to-maturity cases, although the complex logarithm is restricted to the principal branch.