Dynamic Model of Optimal Growth Rate and Payout Ratio:

A Joint Optimization Approach

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Abstract

The main purpose of this paper is to derive a dynamic model which jointly optimizes growth rate and payout ratio. By optimizing stock price per share in terms of negative exponential utility function, we derive a logistic equation which was first derived by Pierre Verhulst (1845 and 1847) to obtain the optimal growth rate. In addition, we generalized Higgins' (1977, 1981, and 2008) sustainable growth by allowing new equity issue. To the best of our knowledge, we have derived a new theoretical model of the optimal growth rate and payout ratio which can be used to identify the specification error of empirical research in determining the payout ratio and growth rate.

Key Words: Dividend Policy; Optimal Payout Ratio; Growth Rate; Specification Error; Logistic Equation
JEL Classification: C10, G35

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1. Introduction

The relationship between the optimal dividend policy and the growth rate have been analyzed at length by Gordon (1962), Lintner (1964), Lerner and Carleton (1966), Modigliani and Miller (1961), Miller and Modigliani (1966), and others. In addition, Higgins (1977, 1981, and 2008) derives a sustainable growth rate assuming that a firm can use retained earnings and issue new debt to finance the growth opportunity of the firm. These authors, though pioneering in their efforts, focus on conducting their analyses at the equilibrium point while the focus of our paper is on analyzing the time path that leads to the equilibrium.

A growing body of empirical literature focuses on the relationship between the optimal dividend payout policy and the growth rate. For example, Rozeff (1982) showed that the optimal dividend payout is related to the fraction of insider holdings, the growth of the firm, and the firm's beta coefficient. Grullon et al. (2002), DeAngelo et al. (2006), and DeAngelo and DeAngelo (2006) suggest that increases in dividends convey information about changes in a firm's life cycle from a higher growth phase to a lower growth phase. Fama and French (2001) find that firms tend to pay dividends when they experience high profitability and low growth rates by using the free cash flow hypothesis. Benartzi et al. (1997) and Grullon et al. (2002) show that dividend changes are related to the change in the growth rate and the change in the rate of return on asset.

The present study deviates from the earlier studies in two major aspects – First, this study develops a fully dynamic model for determining the time optimal growth and dividend policy under stochastic conditions; secondly, the focus is on tracing the time optimal path of the relevant decision variables and exploring their inter-temporal dependencies. The entire analysis in this paper is carried out in the stochastic control theory framework. The focus of the paper is on the rigorous development of a model for maximizing the value of the firm revealing (i) the exact relationships between the dividend policy and the time rate of change in growth, the profit possibility function and their distribution parameters, (ii) the effect of varying time horizons, stochastic initial conditions, and the degree of market perfection in determining an optimal growth and dividend policy for the firm, and (iii) the implications stochasticity, stationarity (in the wide and the strict sense),[1] and nonstationarity have on the rate of return and growth rate for corporate dividend policy decision under uncertainty.

Section 2 develops certain essential elements necessary for the control theory model

[1] Pease see Anderson (1994).

presented in the subsequent sections. The fundamental model expresses a firm's risk adjusted stock price assuming that (i) the new assets of a firm can be financed by new debt, external equity, and internal equity through retained earnings, (ii) the rate of return on equity is stochastic, and (iii) the growth rate varies over time. Section 3 presents the simultaneous solution to the optimization of the growth rate and the outside equity financing. The model is analyzed under the dynamic growth rate assumption and the optimal time path of growth is traced. Section 4 derives the equation for the optimal payout ratio from the model developed in the earlier sections. Initially, we develop the most general form of the optimal payout expression assuming that the rate of return on equity is non-stationarily distributed in the sense that the mean and variance of the rate of return on equity are both functions of time, and the optimal growth rate changes over time. From this general form, models of several other authors are shown to follow as special cases. Section 5 deals with the stochastic growth rate and the identification of specification error introduced in the results by its misspecification as deterministic. This is followed by a short note on the stochasticity of the initial conditions and the final conclusions of the paper.

2. Development of the Model

To explore the optimal payout policy and growth rate, we allow that a firm can finance growth by new debt, external equity, and internal equity through retained earnings, and thus leave the growth unconstrained by retained earnings. Our model has the usual assumptions of rational investor behavior, zero transactions costs, and absence of tax differentials between dividends and capital gains and between distributed and undistributed profits. We further assume that the rate of return on equity is nonstationarily distributed and the growth rate varies over time. The model to maximize price is developed under stochastic growth rate assumptions, but first a simplified case under a deterministic time variant growth rate is presented. Under this specification neither the growth rate nor the level of assets in any time interval are pre-determined. Thus, the asset size at time *t* is

$$A(t) = A(o)e^{\int_o^t g(s)ds}, \tag{1}$$

where $A(o) =$ initial total asset ,

$A(t) =$ total assets at time t ,

$g(t) =$ time variant growth rate,

$s =$ the proxy of time in the integration.

Assuming there exists a constant leverage ratio for a firm, the earnings of this firm can also be defined as a stochastic variable which is the product of the rate of return on equity times the total equity.

$$\begin{aligned}\tilde{Y}(t) &= R\tilde{O}A(t)\,A(o)\,e^{\int_o^t g(s)ds} \\ &= \frac{\tilde{r}'(t)}{1-L}A(o)(1-L)\,e^{\int_o^t g(s)ds} \\ &= \tilde{r}(t)\,A'(o)\,e^{\int_o^t g(s)ds},\end{aligned} \tag{2}$$

where $\tilde{Y}(t) =$ earnings of the leveraged firm at time t,

$R\tilde{O}A(t) =$ the rate of return on total asset for a leverage firm at time t,

$\tilde{r}(t) = \dfrac{R\tilde{O}A(t)}{1-L} =$ the rate of return on total equity at time t, normally distributed with mean $\overline{r}(t)$ and variance $\sigma^2(t)$,

$A'(o) = (1-L)A(o) =$ the total equity at time 0,

L = the debt to total assets ratio.

In the rest of this study, we will use firm's earnings in terms of rate of return on equity and total equity.

In addition, we denote $\dot{n}(t) = dn(t)/dt$, where $n(t)$ is the number of shares of common stock outstanding at time t. We allow that a firm can finance its new investment through retained earnings, external equity, and new debt when it faces a growth opportunity. We also assume that there is a target leverage ratio for a firm by allowing a firm can only issue a proportional debt to meet its target leverage ratio. Therefore, the growth of a firm cannot be only from issuing new debt, but also from retaining its earnings and/or issuing external equity. Then, the change of investment can be defined as

$$\frac{dA(t)}{dt} = \dot{A}(t) = g(t)\,A(o)\,e^{\int_o^t g(s)ds} = \tilde{Y}(t) - \tilde{D}(t) + \lambda\dot{n}(t)\,p(t) + L\dot{A}(t)\,, \qquad (3)$$

where $\tilde{D}(t) =$ the total dollar dividend at time t;

$p(t) =$ price per share at time t;

$\lambda =$ degree of market perfections, $0 < \lambda \leq 1$;

$\lambda\dot{n}(t)\,P(t) =$ the proceeds of new equity issued at time t;

$L =$ the debt to total assets ratio.

Note that the value of λ equal to one indicates that the new shares can be sold by the firm at current market prices.

Equation (3) is a generalized equation which Higgins (1977, 1981, and 2008) used to derive his sustainable growth rate. Higgins' equation allows only internal source and external debt financing. Equation (3) in our model also allows external equity financing.[2]

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[2] The model defined in the equation (3) is for the convenience purpose. If we want to compare our model with Higgin's (1977) sustainable growth rate model, we need to modify equation (3) as follows.

$$\dot{A}(t) = g(t)\,A(o)\,e^{\int_o^t g(s)ds} = \tilde{Y}(t) - \tilde{D}(t) + \lambda\dot{n}(t)\,p(t) + (D/E)\left[\tilde{Y}(t) - \tilde{D}(t) + \lambda\dot{n}(t)\,p(t)\right]$$

(3a).

From (3a), we can obtain $g(t) = \frac{ROE(1-d)}{1-ROE(1-d)} + \frac{\lambda\dot{n}(t)\,p(t)/E}{\left[1-ROE(1-d)\right]}$, which is the generalized version of Higgins' (1977) sustainable growth rate model. Therefore, our model shows that Higgins' (1997) sustainable growth rate is under-estimated due to the omission of the source of the growth related to new equity issue which is the second term of our model.

From equations (2') and (3), we can obtain the dividends per share as

$$\tilde{d}(t) = \frac{\tilde{D}(t)}{n(t)} = \frac{\left[\tilde{r}(t) - g(t)\right] A'(o) e^{\int_o^t g(s)ds} + \lambda \dot{n}(t) p(t)}{n(t)}, \tag{4}$$

and the mean and variance of the dividends per share can be expressed as

$$E\left[\tilde{d}(t)\right] = \frac{\left(\overline{r}(t) - g(t)\right) A'(o) e^{\int_o^t g(s)ds} + \lambda \dot{n}(t) p(t)}{n(t)}$$
$$Var\left[\tilde{d}(t)\right] = \frac{A'(o)^2 \sigma^2(t) e^{2\int_o^t g(s)ds}}{n^2(t)} \tag{5}$$

Also, we postulate a utility function of the following form[3]

$$U\left[\tilde{d}(t)\right] = -e^{-a\tilde{d}(t)}, \text{ where } a > 0. \tag{6}$$

From the certainty equivalent principle and the moment generating technique[4], the certainty equivalent dividend stream can be written as

$$\hat{d}(t) = \frac{\left(\left[\overline{r}(t) - g(t)\right] A'(o) e^{\int_o^t g(s)ds} + \lambda \dot{n}(t) p(t)\right)}{n(t)} - \frac{a' A'(o)^2 \sigma^2(t) e^{2\int_o^t g(s)ds}}{n(t)^2}, \tag{7}$$

where $\hat{d}(t)$ is the certainty equivalent value of $\tilde{d}(t)$ and $a' = a/2$.

Following Lintner (1964), we observe that the stock price should equal the present value of this certainty equivalent dividend stream discounted at the cost of capital, i.e.

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[3] For a detailed analysis of the various utility functions see Pratt (1964). Exponential, hyperbolic, and quadratic forms have been variously used in the literature but the first two seem to have preference over the quadratic form since the latter has the undesirable property that it ultimately turns downwards.

[4] See Simon (1956) and Theil (1957) for the certainty equivalence principle and Hogg et al. (2004) for the moment generating technique.

$$p(o)=\int_0^T \hat{d}(t)e^{-kt}dt \qquad (8)$$

where $p(o)$ = the stock price at the present time ,

k = the cost of capital,

T = the planning horizon.

This is the fundamental model which will be employed in the subsequent sections to derive the functional forms of $n(t)$ and $g(t)$ that simultaneously optimize $p(o)$ and find the optimal growth rate and optimal payout ratio of a firm.

3. Optimal Growth Rate

In this section, we maximize $p(o)$ simultaneously with respect to the growth rate and the number of shares outstanding. To a large extent, the profit possibility function for a firm is exogenously affected by a variety of time variant factors such as factor and product market conditions, although, as Lintner (1964) points out, it conceivably could be affected by its past and present policies, say, with regard to research and development. The decision on the rate of investment in any time interval, however, is largely endogenous to the firm, *ceteris paribus*. Substituting equation (7) into equation (8), we observe

$$p(o)=\int_0^T\left[\frac{\left[\bar{r}(t)-g(t)\right]A'(o)e^{\int_0^t g(s)ds}+\lambda\dot{n}(t)p(t)}{n(t)}-\frac{a'A'(o)^2\sigma^2(t)e^{2\int_0^t g(s)ds}}{n(t)^2}\right]e^{-kt}dt. \qquad (9)$$

To maximize equation (9), we observe that

$$p(t) = \int_t^T \hat{d}(s) e^{-k(s-t)} ds = e^{kt} \int_t^T \hat{d}(s) e^{-ks} ds. \tag{10}$$

From equation (10), we can formulate a differential equation as

$$\frac{dp(t)}{dt} = \dot{p}(t) = kp(t) - \hat{d}'(t). \tag{11}$$

Substituting equation (7) into equation (11), we have a differential equation as

$$\dot{p}(t) + \left[\frac{\lambda \dot{n}(t)}{n(t)} - k \right] p(t) = -H(t), \tag{12}$$

where

$$H(t) = \frac{\left(\bar{r}(t) - g(t)\right) A'(o) e^{\int_0^t g(s)ds}}{n(t)} - \frac{a' A'(o)^2 \sigma^2(t) e^{2\int_0^t g(s)ds}}{n(t)^2}. \tag{13}$$

Solving the differential equation (12)[5], we have

$$p(t) = \frac{e^{kt}}{n(t)^\lambda} \int_t^T H(s) n(s)^\lambda e^{-ks} ds. \tag{14}$$

Then, equation (15) can be obtained from equations (13) and (14) implying that the initial value of a stock can be expressed as the summation of present values of its earnings stream adjusted by the risk taken by the firm.

$$p(o) = \frac{1}{n(o)^\lambda} \int_0^T \left[n(t)^{\lambda-1} \left(\bar{r}(t) - g(t)\right) A'(o) e^{\int_0^t g(s)ds} - a' A'(o)^2 \sigma^2(t) n(t)^{\lambda-2} e^{2\int_0^t g(s)ds} \right] e^{-kt} dt \tag{15}$$

[5] For the derivation of the partial differential equation, please refer to the Appendix A.

We maximize the stock price, $p(o)$, defined in equation (15) by allowing both growth rate $g(t)$ and number of shares outstanding $n(t)$ change over time. When the growth rate $g(t)$ is determined, then the amount of new investment can also be obtained. Moreover, once the amount of new investment and new equity issue $\dot{n}(t)p(t)$ are determined, then we can calculate the amount of the new debt and retained earnings (or dividend payout) at the same time. Therefore, it implies that we can obtain the optimal growth rate and optimal payout ratio simultaneously by maximizing the stock price[6].

The Euler-Lagrange condition for the optimization of $p(o)$ is given by equations (16) and (17).[7]

$$n(t)=\frac{(\lambda-2)a'A'(o)\sigma^2(t)e^{\int_0^t g(s)ds}}{(\lambda-1)\left[\bar{r}(t)-g(t)\right]}, \text{ and} \tag{16}$$

$$n(t)=\frac{2a'A'(o)\sigma^2(t)e^{\int_0^t g(s)ds}}{\bar{r}(t)-g(t)-\dfrac{\dot{g}(t)}{g(t)}}. \tag{17}$$

From equation (16) and (17), we have

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[6] For example, we assume that the amount of new investment is \$1,000,000, new equity issue is \$200,000, the target debt to total asset ratio is 30%, and net income is \$2,000,000. Then we know the new debt issue is \$1,000,000 x 30% = \$300,000. Therefore, the retained earnings will be \$1,000,000 - \$200,000, - \$300,000 = \$500,000. In other words, the optimal payout ratio is (\$2,000,000 - \$500,000)/\$2,000,000 = 75%.

[7] To the best of our knowledge, this is the first logistic equation derived in finance research. For the detail derivation of equations (16) and (17), please see Appendix B.

$$\lambda g(t)^2 - \lambda \bar{r}(t) g(t) + (2-\lambda)\dot{g}(t) = 0. \tag{18}$$

Under the assumption of $\bar{r}(t) = \bar{r}$, equation (18) is a non-linear differential equation in $g(t)$, which is the logistic equation discovered by Pierre Verhulst (1845 and 1847). Therefore, the optimal growth rate $g^*(t)$ from equation (18) is[8]

$$\begin{aligned} g^*(t) &= \frac{\bar{r}}{1-\left(1-\frac{\bar{r}}{g_o}\right)e^{\lambda \bar{r}t/(\lambda-2)}} \\ &= \frac{g_o \bar{r}}{g_o + (\bar{r}-g_o)e^{\lambda \bar{r}t/(\lambda-2)}} \end{aligned}, \tag{19}$$

where g_o is the initial growth rate.

Equation (19) shows that there exists an optimal growth rate, $g^*(t)$, when we maximize a firm's stock price per share by jointly determining the growth rate and the optimal payout ratio. Therefore, different from Higgins' (1977) sustainable growth rate model, the payout ratio (or retention rate) does not affect the optimal growth rate.

The optimal growth rate can be determined by (i) the time horizon (t), (ii) the degree of market perfection or imperfection (λ), (iii) the rate of return on equity $(\bar{r})$, and (iv) the initial growth rate (g_o). In addition, there is a special characteristic of the logistic equation called the "*mean-reverting process*". In the following sub-sections, we will numerically and graphically show how the mean-reverting process of the optimal

[8] For the detail derivation of Equation (19), please see Appendix C.

growth rate works. Furthermore, we provide the sensitivity analysis and the partial derivatives on the optimal growth rate with respect to (i) the time horizon (t), (ii) the degree of market perfection (λ), (iii) the rate of return on equity $(\bar{r})$, and (iv) the initial growth rate (g_o). We will discuss how these factors affect the optimal growth rate respectively.

3.1 Case I: Time Horizon

Taking the derivative of the optimal growth rate with respect to the time, we can obtain equation (20).

$$\dot{g}^*(t) = \frac{\partial g^*(t)}{\partial t} = \frac{\bar{r}\left(1-\frac{\bar{r}}{g_o}\right)\left(\frac{\bar{r}\lambda}{(\lambda-2)}\right)e^{\bar{r}\lambda t/(\lambda-2)}}{\left[1-\left(1-\frac{\bar{r}}{g_o}\right)e^{\bar{r}\lambda t/(\lambda-2)}\right]^2}, \tag{20}$$

where $g^*(t) = \dfrac{\bar{r}}{1-\left(1-\frac{\bar{r}}{g_o}\right)e^{\lambda\bar{r}t/(\lambda-2)}}$.

From equation (20), we can find that the optimal growth rate follows a mean-reverting process. If the initial growth rate, (g_0), is greater than the rate of return on equity, $(\bar{r})$, the change of the optimal growth rate, $\dot{g}^*(t)$, is negative. If the initial growth rate is lower than the rate of return on equity, the change of the optimal growth rate is positive. If the initial growth rate is equal to the rate of return on equity, the change of the optimal growth rate is zero. Therefore, the optimal growth rate is approaching to the rate of return on equity, $\bar{r}$, over time. Table 1 and Figure 1 show the time path of the optimal growth rate from the initial conditions (g_o) to its steady state value.

<Insert Table 1>

<Insert Figure 1>

The important implications of the results derived from equation (18) and partially graphed in Figure 1 are as follows:

(i) Under a finite time horizon, $\begin{array}{l} g^*(t) = \bar{r} \text{ and } \dot{g}^*(t) = 0 \text{ when } g_o = \bar{r} \\ g^*(t) < \bar{r} \text{ and } \dot{g}^*(t) < 0 \text{ when } g_o < \bar{r} \\ g^*(t) > \bar{r} \text{ and } \dot{g}^*(t) > 0 \text{ when } g_o > \bar{r} \end{array}$.

(ii) When time horizon approaches infinity, the optimal growth rate approaches the steady state value of $\bar{r}$.

(iii) The greater the degree of imperfection in the market, as indicated by a lower value of λ, $(0 < \lambda < 1)$, the slower the speed with which the firm attains the steady state optimal growth rate. One feasible reason is that the firm can issue additional equity only at a price lower than the current market price indicated by the value of $\lambda < 1$.

3.2 Case II: Degree of Market Perfection

Taking the derivative of the optimal growth rate with respect to the degree of market perfection, λ , we can obtain equation (21).

$$\frac{\partial g^*(t)}{\partial \lambda} = \frac{\left(\frac{\bar{r}}{g_o}-1\right)e^{\bar{r}\lambda t/(\lambda-2)}\frac{2\bar{r}t}{(\lambda-2)^2}}{\left[1-\left(1-\frac{\bar{r}}{g_o}\right)e^{\bar{r}\lambda t/(\lambda-2)}\right]^2}, \tag{21}$$

where $g^*(t) = \dfrac{\bar{r}}{1-\left(1-\frac{\bar{r}}{g_o}\right)e^{\lambda\bar{r}t/(\lambda-2)}}$.

We find that the sign of equation (21) depends on the sign of $\left(\frac{\bar{r}}{g_o}-1\right)$. If the initial growth rate is less than the rate of return on equity, equation (21) is positive. Therefore, the optimal payout ratio at time *t* tends to be closer to the rate of return on equity if the degree of market perfection increases. Comparing Figure 1.a and Figure 1.c to Figure 1.b and Figure 1.d, we can find when the degree of market perfection is 0.95, the optimal growth rate will converge to its target rate (rate of return on equity) after 20 years. When the degree of market perfection is 0.60, the optimal growth rate cannot converge to its target rate (rate of return on equity) even after 30 year. Figure 2 shows the optimal growth rates of different rates of return on equity and degrees of market perfection when the initial growth rate is 5 percent and the time is 3. We find that the more perfect the market is, the faster the optimal growth rate adjusts. Therefore, the mean-reverting process of the optimal growth rate is faster if the market is more perfect.

<Insert Figure 2>

3.3 Case III: Rate of Return on Equity

Taking the derivative of the optimal growth rate with respect to the rate of return on equity, $\overline{r}$, we can obtain equation (22).

$$\frac{\partial g^*(t)}{\partial \overline{r}} = \frac{g_o\left\{\left[g_o\left(1-e^{\lambda\overline{r}t/(\lambda-2)}\right)\right]+\left[(g_o-\overline{r})\frac{\lambda t}{(\lambda-2)}\overline{r}e^{\lambda\overline{r}t/(\lambda-2)}\right]\right\}}{\left[g_o+(\overline{r}-g_o)e^{\lambda\overline{r}t/(\lambda-2)}\right]^2}, \quad (22)$$

where $g^*(t) = \frac{g_o\overline{r}}{g_o-(\overline{r}-g_o)e^{\lambda\overline{r}t/(\lambda-2)}}$.

Because the degree of market perfection (λ) is between zero and one, $\frac{\lambda t}{(\lambda-2)}$ is negative and $e^{\lambda\overline{r}t/(\lambda-2)}$ is between zero and one. Therefore, the sign of equation (22) is positive when the initial growth rate is less than the rate return on equity. That is, in order to catch up to the higher target rate, a firm should increase its optimal growth rate. We further provide a sensitivity analysis to investigate the relationship between the change in the optimal growth rate and the change in the rate of return on equity. Panel A and C of Table 1 and Figure 1.a and Figure 1.c present the mean-reverting processes of the optimal growth rate when the rate of return on equity is 40 percent. Panel B and Panel D of Table 1 and Figure 1.b and Figure 1.d present those when the rate of return on equity is 50 percent. We find that, holding the other factors constant, the optimal growth rate under a 50 percent rate of return on equity is higher than the optimal growth rate under a 40 percent rate of return on equity. Figure 2 shows that, under different degrees of market perfection, the optimal growth rates increase when the rate of return on equity increases. Therefore, when the rate of return on equity increases, the optimal growth rates increase under different conditions of initial growth rates, time horizons, and degrees of market perfections.

3.4 Case IV: Initial Growth Rate

Taking the derivative of the optimal growth rate with respect to the initial growth rate, g_0, we can obtain equation (23).

$$\frac{\partial g^*(t)}{\partial g_o} = \frac{\left(\frac{\bar{r}}{g_o}\right)^2 e^{\lambda \bar{r} t/(\lambda-2)}}{\left[1-\left(1-\frac{\bar{r}}{g_o}\right)e^{\lambda \bar{r} t/(\lambda-2)}\right]^2}, \tag{23}$$

where $g^*(t) = \frac{\bar{r}}{1-\left(1-\frac{\bar{r}}{g_o}\right)e^{\lambda \bar{r} t/(\lambda-2)}}$.

We can easily determine that the sign of equation (23) is positive. From Table 1, we can also find the higher optimal growth rate if the initial growth rate is higher. Moreover, Figure 1 shows that the line of the optimal growth rate over time does not cross over each other, meaning that the mean-reverting process of the optimal growth rate under an initial growth rate far from the objective value cannot be faster than that under an initial growth rate closer to the objective value.

4. Optimal Dividend Policy

In this section, we address ourselves to the problem of an optimum dividend policy for the firm. Using the model developed in the earlier sections, we maximize $p(o)$ simultaneously with respect to the growth rate and the number of shares outstanding. We can derive a general form for the optimal payout ratio as following.[9]

$$\frac{\bar{D}(t)}{\bar{Y}(t)} = \left[1 - \frac{g^*(t)}{\bar{r}(t)}\right] \cdot \left[1 + \left[e^{kt - \lambda\int_0^t g^*(s)ds}\right] \frac{\lambda\left[\left(\dot{\sigma}^2(t) + \sigma^2(t) g^*(t)\right)\left(\bar{r} - g^*(t)\right) + \sigma^2(t)\dot{g}^*(t)\right]}{(2-\lambda)\sigma^2(t)^{\lambda}\left(\bar{r} - g^*(t)\right)^{3-\lambda}} W\right] \tag{29}$$

where $W = \int_t^T e^{\lambda\int_0^s g^*(u)du - ks} \sigma^2(s)^{\lambda - 1}\left(\bar{r} - g^*(s)\right)^{2-\lambda} ds$.

Equation (29) implies that the optimal payout ratio of a firm is a function of the optimal growth rate $g^*(t)$, the change in optimal growth rate $\dot{g}^*(t)$, the rate of return on equity $\bar{r}$, the degree of market perfection λ, the cost of capital k, the risk of the rate of return on equity $\sigma^2(t)$, and the change in the risk of rate of the return on equity $\dot{\sigma}^2(t)$. Previous empirical studies find that a firm's dividend policy is related to the firm's risk, growth rate, and proxies of profitability, such as the rate of return on assets and the market-to-book ratio. (eg. Rozeff(1982), Benartzi et al. (1997), Grullon et al. (2002), Fama and French (2001), DeAngelo et al. (2006), and Denis and Osobov (2008).) However, these studies omit some important factors such as the degree of market

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[9] For the detailed derivation of Equation (29), please see the Appendix D.

perfection, the cost of capital, the rate of return on equity, the change in the growth rate, and the change in risk. Therefore, our theoretical model implies that previous studies might have used misspecified models to conduct their empirical research.

5. Stochastic Growth Rate

5.1 Stochastic Growth Rate and Specification Error

This section attempts to identify the possible error in the optimal dividend policy introduced by misspecification of the growth rate as deterministic. Lintner (1964) explicitly introduced uncertainty of growth in his valuation model. The uncertainty is shown to have two components – one derived from the uncertainty of the rate of return and, second, from the increase in this uncertainty as a linear function of futurity. In the present analysis we introduced a fully dynamic time variant growth rate to clearly see the impact of its stochasticity on the optimal dividend policy.

With the stochastic rate of return on equity, $\tilde{r}(t) \sim N\left(\bar{r}(t), \sigma^2(t)\right)$, the asset size and the growth rate for any time interval also become stochastic. This can be seen more clearly by rewriting equation (3) explicitly in the growth form as follow:

$$\tilde{g}(t) = \frac{\dot{A}(t)}{A(t)} = b\tilde{r}(t) + \frac{\lambda \dot{n}(t)\, p(t)}{(1-L)\, A(t)}, \tag{30}$$

where b is the retention rate and $(1-L)\, A(t)$ is the total equity at time t. Therefore, the growth rate of a firm is related not only to how much earnings it retains, but also to how many new shares it issues.

Thus, with a stochastic rate of return, the growth rate in equation (30) also becomes stochastic, $\tilde{g}(t) \sim N\left(g(t), \sigma_g^2(t)\right)$. A further element of stochasticity in $\tilde{g}(t)$ is introduced by the stochastic nature of λ indicating the uncertainty about the price at which new equity can be issued relative to *p*(*t*). Misspecification of the growth rate as deterministic in the earlier sections introduced an error in the optimal dividend policy, which we now proceed to identify. Reviewing equations (1) to (4) under a stochastic growth rate specification, we derive from equation (4) the expression equation (31) as follows:

$$\tilde{d}(t) = \frac{\tilde{D}(t)}{n(t)} = \frac{\left[\tilde{r}(t) - \tilde{g}(t)\right] A'(o) e^{\int_0^t \tilde{g}(s)\ ds} + \lambda \dot{n}(t) p(t)}{n(t)} \tag{31}$$

It may be interesting to compare this model with the internal growth models. Notice that if the external equity financing is not permissible, the growth rate, equation (30), is equal to the retention rate times the profitability rate, as in Lintner (1964) and Gordon (1963). Also, notice that under the no external equity financing assumption, equation (31) reduces to $\tilde{d}(t) = \tilde{d}_o\ e^{\int_0^t \tilde{g}(s)\ ds}$, which is Lintner's equation (8) with the only difference that, in our case, the growth rate is an explicit function of time.

To carry the analysis further under a time variant stochastic growth rate, we derive the mean and variance of $\tilde{d}(t)$ as follows:

$$E\left[\tilde{d}(t)\right]=\frac{\left[\left(\bar{r}(t)-g(t)\right)A'(o)e^{\int_0^t g(s)ds}+\lambda\dot{n}(t)p(t)\right]}{n(t)} -\frac{Cov\left(\tilde{r}(t),A'(o)e^{\int_0^t \tilde{g}(s)ds}\right)+E\left[\tilde{\varepsilon}(t)e^{\int_0^t \tilde{g}(s)ds}\right]}{n(t)} \tag{32a}$$

and

$$\text{Var}\left[\tilde{d}(t)\right]=\frac{A'(o)^2\sigma^2(t)e^{2\int_0^t \tilde{g}(s)ds}}{n(t)}+\frac{A'(o)^2\,\text{Var}\left[\tilde{\varepsilon}(t)e^{\int_0^t \tilde{g}(s)ds}\right]}{n(t)^2}, \tag{32b}$$

where $\tilde{\varepsilon}(t)=\tilde{g}(t)-g(t)$.

Comparing equation (32) and equation (5), we observe that both the mean and the variance of $\tilde{d}(t)$ will be subject to specification error if the deterministic growth rate is inappropriately employed. Equation (32a) implies that the numerator in the equation for the optimal payout ratio has the following additional element:

$$\frac{-\text{Cov}\left(\tilde{r}(t),A(o)e^{\int_0^t \tilde{g}(s)ds}\right)+E\left(\tilde{\varepsilon}(t)e^{\int_0^t \tilde{g}(s)ds}\right)}{n(t)} \tag{33}$$

Now notice that the second term in the expression of equation (33) tends to be dominated more and more by the first term as the time *t* approaches to infinity.[10]

Thus, the direction of error introduced in the optimal payout ratio when the growth rate is arbitrarily assumed to be non-stochastic increasingly depends upon the size of the covariance term in equation (33). If the covariance between the rate of return on equity and the growth rate is positive, the optimal payout ratio under the stochastic growth rate assumption would be lower even if the rate of return for the firm increases over time. In other words, the payout ratio in section 4 under the deterministic growth rate assumption is an over-estimate of the optimal payout ratio. On the other hand, if the covariance between the rate of return on equity and the growth rate is negative (that is, the higher the growth rate, the lower the rate of return on equity), the payout ratio derived in the preceding section is an under-estimate of the optimal payout ratio.

5.2 Stochastic Initial Conditions

At this point a note on the stochasticity of the initial conditions and the significance of the time path of the growth rate is also in order. From the specification of the

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[10] From Goldberger (1966), we know that $\operatorname*{plim}_{s\to\infty}\left[\tilde{\varepsilon}(s)e^{\int_0^t \tilde{g}(s)ds}\right] = \operatorname*{plim}_{s\to\infty}\left[\tilde{\varepsilon}(s)\right]\operatorname*{plim}_{s\to\infty}\left[e^{\int_0^t \tilde{g}(s)ds}\right]$ (A)

From Astrom (2006), we also know that

$$\operatorname*{plim}_{s\to\infty}\left[e^{\int_0^t \tilde{g}(s)ds}\right] = e^{\int_0^t g(s)ds} \qquad \text{(B)}$$

Substituting (B) into (A), we have

$$\operatorname*{plim}_{s\to\infty}\left[\tilde{\varepsilon}(s)e^{\int_0^t g(\tilde{s})ds}\right] = e^{\int_0^t g(s)dt}\operatorname*{plim}_{s\to\infty}\left[\tilde{\varepsilon}(s)\right] = 0$$

stochastic growth rate in the previous sub-section, it can be argued that the initial growth rate $\left(g_o\right)$ in equation (19) and illustrated in Figure 1 is chosen by the management taking into account the uncertainty of the time path of return and its distribution parameters $\left(\bar{r}\left(t\right), \sigma^2\left(t\right)\right)$. Therefore, the determination of the initial growth rate is also stochastic, implying that $g_o = \bar{r}$ can happen only in the expectation sense. Since $g_o = \bar{r}$ is only one point from the entire distribution, there is only a small probability that the steady value of the optimal growth rate happens and that the optimal growth rate does not change over time. This further brings out the importance of analyzing the time path of the growth rate as it approaches the steady state value from the initia1 stochastic conditions. We feel that significant progress in financial decision making based on the detailed analysis of equilibrium values of the underlying parameters has been made in the literature, thanks to the pioneering efforts by Modigliani and Miller (1961), Gordon (1962), and Lintner (1964), to name only a few. Further research, however, needs to be done in tracing the time path leading to these steady state values of parameters under uncertainty.

6. Summary and Conclusion Remarks

Our concern in this paper has been to sharpen the focus on corporate decision rules concerning the optimal growth and dividend payout policies. The inter-dependence of the corporate decisions on these two important policy variables is explicitly recognized in deriving the simultaneous solution to the problem of maximizing the present value of the firm. In more detail, we examine the time path of the optimal growth rate, starting from the stochastic initial conditions, and approaching its steady state value. Important insights into the effects of the degree of market perfection, rate of return on equity, and initial growth rate on the optimal growth rate are also developed. Furthermore, we derive the optimal payout policy under most genera1 conditions with regard to the market imperfection, non-stationarity, and stochasticity of the growth rate and the profitability rate. In particular, the error introduced in the optimal payout ratio by misspecification of the growth rate as deterministic is identified and shown to relate to its initial stochastic conditions. Finally, the model developed in this paper brings out explicitly the influence of riskiness of a non-stationary profitability rate on the optimal dividend policy. Important work, as mentioned earlier, in determining the optimal equilibrium values of corporate decision variables has been done by Gordon (1962), Lintner (1964), Modigliani and Miller (1961), Miller and Modigliani (1966), DeAngelo et al. (1996), Fama and French (2001), and DeAngelo and DeAngelo (2006). However, the need for further research in tracing the optimal time path under uncertainty leading to these equilibrium values and the relevant stability conditions remains and the present paper is a modest step forward in that direction.

In the future research, we will develop the whole results in terms of equation (3a) in footnote 2 instead of equation (3).

Appendix A. Derivation of Equation (14)

This appendix presents a detailed derivation of the solution to the variable partial differential equation, equation (12), which is similar to Gould's (1968) equation (9) in investigating the adjustment cost. Following Gould's (1968) approach, we first derive a general solution for a standard variable partial differential equation. Then we apply this general equation to solve equation (12). The standard variable partial differential equation can be defined as:

$$\dot{p}(t) + g(t)\,p(t) = q(t) \tag{A.1}$$

As a particular case of equation (A.1), the equation

$$\dot{p}(t) + g(t)\,p(t) = 0 \quad \text{or} \quad \frac{\dot{p}(t)}{p(t)} = -g(t) \tag{A.2}$$

has a solution

$$p(t) = c \cdot \exp\left(-\int g(t)dt\right). \tag{A.3}$$

By substituting constant c with function $c(t)$, we have the potential solution to equation (A.1)

$$p(t) = c(t) \cdot \exp\left(-\int g(t)dt\right). \tag{A.4}$$

Taking a differential with respect to t, we obtain:

$$\begin{aligned}\dot{p}(t) &= \dot{c}(t)\cdot\exp\left(-\int g(t)dt\right) - c(t)\cdot\exp\left(-\int g(t)dt\right)g(t) \\ &= \dot{c}(t)\cdot\exp\left(-\int g(t)dt\right) - p(t)g(t)\end{aligned} \tag{A.5}$$

Therefore,

$$\dot{p}(t) + p(t)g(t) = \dot{c}(t)\cdot\exp\left(-\int g(t)dt\right). \tag{A.6}$$

From equations (A.1) and (A.6), we have

$$\dot{c}(t)\cdot\exp\left(-\int g(t)dt\right) = q(t). \tag{A.7}$$

Equivalently,

$$\dot{c}(t) = q(t)\cdot\exp\left(\int g(t)dt\right). \tag{A.8}$$

Therefore,

$$c(t) = \int q(t)\cdot\exp\left(\int g(t)dt\right)dt. \tag{A.9}$$

Substituting equation (A.9) into equation (A.3), we have the general solution of equation (A.1),

$$p(t) = \exp\left(-\int g(t)dt\right)\cdot\left[\int q(t)\exp\left(\int g(t)dt\right)dt\right]. \tag{A.10}$$

To solve equation (17), we will apply the above result. Let $g(t) = \lambda\frac{\dot{n}(t)}{n(t)} - k$ and $q(t) = -H(t)$.

Since

$$
\begin{aligned}
\exp\left(\int g(t)dt\right) &= \exp\left(\int\left(\lambda\frac{\dot{n}(t)}{n(t)}-k\right)dt\right) \\
&= \exp\left(\lambda\int\frac{\dot{n}(t)}{n(t)}dt-kt\right) \\
&= \exp\left(\lambda\ln(n(t))-kt+c\right) \qquad \text{(A.11)} \\
&= c_1\cdot n(t)^{\lambda}\exp(-kt),
\end{aligned}
$$

where $c_1 > 0$.

Then we have,

$$P(t) = c_2\cdot n(t)^{-\lambda}\exp(kt)\cdot\left[\int q(t)c_3\cdot n(t)^{\lambda}\exp(-kt)dt\right], \qquad \text{(A.12)}$$

where $c_2 > 0$, and $c_3 > 0$

or equivalently,

$$P(t) = c_4\cdot\frac{\mathrm{e}^{kt}}{n(t)^{\delta}}\cdot\left[\int -H(t)\frac{n(t)^{\lambda}}{e^{kt}}dt\right], \qquad \text{(A.13)}$$

where $c_4 > 0$.

Finally, we have

$$P(t) = c\frac{\mathrm{e}^{kt}}{n(t)^{\lambda}}\cdot\int H(t)n(t)^{\lambda}e^{-kt}dt, \qquad \text{(A.14)}$$

Changing from an indefinite integral to a definite integral, equation (A.13) can be shown as

$$p(t) = \frac{e^{kt}}{n(t)^{\lambda}}\int_t^T H(s)n(s)^{\lambda}e^{-ks}ds,$$

which is equation (14).

APPENDIX B. Derivation of Equation (18)

This appendix presents a detailed procedure for deriving equation (18). Following Euler-Lagrange condition [see Chiang, (1984)], we first take the first order condition on equation (15) with respect to t allowing only $n(t)$ and $g(t)$ to change over time, and set the first order condition equal to zero. Then we obtain

$$\begin{aligned}\frac{\partial p(o)}{\partial t} &= (\lambda-1)n(t)^{\lambda-2}\,\dot{n}(t)\left[r(t)-g(t)\right]A'(o)\exp\left(\int_0^t g(s)\right)ds \\ &\quad - a'A'^2(o)\sigma^2(t)(\lambda-2)n(t)^{\lambda-3}\,\dot{n}(t)\exp\left(2\int_0^t g(s)ds\right)=0\end{aligned} \tag{B.1}$$

$$\begin{aligned}\frac{\partial p(o)}{\partial t} &= \Big[n(t)^{\lambda-1}\,r(t)A'(o)g(t)\exp\left(\int_0^t g(s)ds\right) \\ &\quad - n(t)^{\lambda-1}\,A'(o)\dot{g}(t)\exp\left(\int_0^t g(s)ds\right) \\ &\quad - n(t)^{\lambda-1}\,A'(o)g(t)^2\exp\left(\int_0^t g(s)ds\right)\Big] \\ &\quad - \left[2a'A'^2(o)\sigma^2(t)n(t)^{\lambda-2}\,g(t)\exp\left(2\int_0^t g(s)ds\right)\right]=0\end{aligned} \tag{B.2}$$

After rearranging equations (B.1) and (B.2), we can obtain equations (16) and (17).

$$n(t)=\frac{(\lambda-2)a'A'(o)\sigma^2(t)\exp\left(\int_0^t g(s)ds\right)}{(\lambda-1)\left[r(t)-g(t)\right]} \tag{16}$$

$$n(t)=\frac{2a'A'(o)\sigma^2(t)\exp\left(\int_0^t g(s)ds\right)}{r(t)-g(t)-\dfrac{\dot{g}(t)}{g(t)}} \tag{17}$$

Therefore, from equation (16) and (17), we can derive the following well-known logistic differential equation as

$$\lambda g(t)^2 - \lambda r(t) g(t) + (2-\lambda)\dot{g}(t) = 0 . \qquad (18)$$

APPENDIX C. Derivation of Equation (19)

This appendix presents a detailed derivation of the solution to the equation (19). To solve partial differential equation, equation (18), which is a logistic equation (or Verhulst model) first published by Pierre Verhulst (1845 and 1847). To solve the logistic equation, we first rewrite equation (18) as a standard form of logistic equation.

$$\frac{dg(t)}{dt} = Ag(t)\left(1 - \frac{g(t)}{B}\right). \tag{C.1}$$

where

$$A = \frac{\lambda\bar{r}}{(2-\lambda)} \quad \text{and} \quad B = \bar{r}\,.$$

Using separation of variables to separate *g* and *t*:

$$\begin{aligned} Adt &= \frac{dg(t)}{g(t)\left(1-\frac{g(t)}{B}\right)} \\ &= \frac{dg(t)}{g(t)} + \frac{\frac{dg(t)}{B}}{\left(1-\frac{g(t)}{B}\right)} \end{aligned} \tag{C.2}$$

We next integrate both sides of equation (C.2),

$$At + C = \ln\left(g(t)\right) - \ln\left(1 - \frac{g(t)}{B}\right). \tag{C.3}$$

Taking the exponential on both sides of equation (C.3),

$$C \cdot \exp(At) = \frac{g(t)}{1 - \frac{g(t)}{B}}. \tag{C.4}$$

When $t=0$, $g(t)=g_0$ and equation (C.4) can be written as

$$C=\frac{g_0}{1-\frac{g_0}{B}}=\frac{Bg_0}{B-g_0}. \tag{C.5}$$

Substitute equation (C.5) into equation (C.4),

$$\frac{Bg_0}{B-g_0}\cdot\exp(At)=\frac{g(t)}{1-\frac{g(t)}{B}}. \tag{C.6}$$

Solving for $g(t)$,

$$\begin{aligned} g(t)&=\frac{\frac{Bg_0}{B-g_0}\exp(At)}{1+\frac{\frac{Bg_0}{B-g_0}\exp(At)}{B}} \\ &=\frac{Bg_0\exp(At)}{B-g_0+g_0\exp(At)} \\ &=\frac{Bg_0}{g_0+(B-g_0)\exp(-At)} \end{aligned} \tag{C.7}$$

Substitute $A=\frac{\lambda\bar{r}}{(2-\lambda)}$ and $B=\bar{r}$ into equation (C.7), we finally can get the solution of equation (18),

$$g(t)=\frac{\frac{\bar{r}}{(1-L)}}{1+\left(\frac{\bar{r}}{g_0}-1\right)\exp\left(\frac{\lambda\bar{r}t}{\lambda-2}\right)},$$

which is equation (19).

APPENDIX D. Derivation of Equation (29)

This appendix presents a detailed derivation of equation (29). To obtain the expression for optimal $n(t)$, given $\bar{r}(t)=\bar{r}$, we substitute equation (19) into equation (16) and obtain

$$n(t)=\frac{(2-\lambda)a'A'(o)\sigma^2(t)e^{\int_0^t g^*(s)ds}}{(1-\lambda)\left(\bar{r}-g^*(t)\right)}. \tag{D.1}$$

From equations (13), (16), and (19), we have

$$H(t)=\frac{\left(\bar{r}-g^*(t)\right)^2(1-\lambda)}{a'\sigma^2(t)(2-\lambda)^2}. \tag{D.2}$$

Substituting equations (D.1) and (D.2) into equation (14) we obtain

$$p(t)=\left(\frac{e^{kt}}{n(t)^{\lambda}}\frac{1-\lambda}{a'(2-\lambda)^2}\int_t^T\frac{n(s)^{\lambda}}{\sigma^2(s)}\left(\bar{r}-g^*(s)\right)^2e^{-ks}ds\right)\quad\text{or}$$

$$p(t)=\left[\frac{e^{kt-\lambda\int_0^t g^*(s)ds}(1-\lambda)\left(\bar{r}-g^*(t)\right)^{\lambda}}{\sigma^2(t)^{\lambda}a'(2-\lambda)^2}\right]W, \tag{D.3}$$

where $W=\int_t^T e^{\lambda\int_0^s g^*(u)du-ks}\sigma^2(s)^{\lambda-1}\left(\bar{r}-g^*(s)\right)^{2-\lambda}ds$.

From equation (D.1), we have

$$\dot{n}(t)=\frac{(2-\lambda)a'A'(o)\dot{\sigma}^2(t)e^{\int_0^t g^*(s)ds}+(2-\lambda)a'A(o)\sigma^2(t)g^*(t)e^{\int_0^t g^*(s)ds}}{(1-\lambda)\left(\bar{r}-g^*(t)\right)} + \frac{(2-\lambda)a'A'(o)\sigma^2(t)e^{\int_0^t g^*(s)ds}\dot{g}^*(t)}{(1-\lambda)\left(\bar{r}-g^*(t)\right)^2}. \tag{D.4}$$

From equations (D.3) and (D.4), we have the amount generated from the new equity issue

$$\dot{n}(t)\,p(t)=\frac{A(o)}{\sigma^2(t)^{\lambda}(2-\lambda)}\Big[\big(\dot{\sigma}^2(t)+\sigma^2(t)g^*(t)\big)\big(\bar{r}-g^*(t)\big)+\sigma^2(t)\dot{g}^*(t)\Big]\cdot \left[e^{kt-(\lambda-1)\int_0^t g^*(s)ds}\big(\bar{r}-g^*(t)\big)^{\lambda-2}\right]W \tag{D.5}$$

From equations (1), (4), and (D.5), we can obtain $\bar{D}(t)=n(t)\bar{d}(t)$ and $\bar{Y}(t)=\bar{r}(t)A(o)e^{\int_0^t g(s)ds}$. Therefore, the optimal payout ratio can be written as

$$\frac{\bar{D}(t)}{\bar{Y}(t)}=\left[1-\frac{g^*(t)}{\bar{r}(t)}\right]\cdot \left[1+\left[e^{kt-\lambda\int_0^t g^*(s)ds}\right]\frac{\lambda\Big[\big(\dot{\sigma}^2(t)+\sigma^2(t)g^*(t)\big)\big(\bar{r}-g^*(t)\big)+\sigma^2(t)\dot{g}^*(t)\Big]}{(2-\lambda)\sigma^2(t)^{\lambda}\big(\bar{r}-g^*(t)\big)^{3-\lambda}}W\right] \tag{29}$$

where $W=\int_t^T e^{\lambda\int_0^s g^*(u)du-ks}\sigma^2(s)^{\lambda-1}\big(\bar{r}-g^*(s)\big)^{2-\lambda}ds$.

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Table 1. Mean-reverting process of the optimal growth rate

This table shows the mean-reverting process of the optimal growth rate for thirty years. Panel A presents, when the rate of return on equity is equal to 40 percent and the degree of market perfection is equal to 0.95, the values of the optimal growth rate under different settings of the initial growth rate. Panel B presents the values of the optimal growth rate when the rate of return on equity is equal to 40 percent and the degree of market perfection is equal to 0.60. Panel C presents the values of the optimal growth rate when the rate of return on equity is equal to 50 percent and the degree of market perfection is equal to 0.95. Panel D presents the values of the optimal growth rate when the rate of return on equity is equal to 50 percent and the degree of market perfection is equal to 0.60

Panel A.	**ROE = 40%** λ **=0.95**							
Time (*t*)	$g_0 = $ **5%**	$g_0 = $ **10%**	$g_0 = $ **15%**	$g_0 = $ **20%**	$g_0 = $ **25%**	$g_0 = $ **30%**	$g_0 = $ **35%**	$g_0 = $ **40%**
1	0.0681	0.1295	0.1851	0.2358	0.2821	0.3246	0.3638	0.4000
2	0.0910	0.1630	0.2212	0.2694	0.3099	0.3443	0.3741	0.4000
3	0.1189	0.1987	0.2560	0.2990	0.3326	0.3595	0.3816	0.4000
4	0.1512	0.2346	0.2874	0.3239	0.3505	0.3709	0.3870	0.4000
5	0.1864	0.2682	0.3142	0.3437	0.3642	0.3793	0.3909	0.4000
6	0.2225	0.2981	0.3361	0.3591	0.3744	0.3854	0.3936	0.4000
7	0.2571	0.3231	0.3533	0.3706	0.3818	0.3897	0.3955	0.4000
8	0.2884	0.3431	0.3663	0.3790	0.3872	0.3928	0.3969	0.4000
9	0.3151	0.3586	0.3759	0.3852	0.3910	0.3949	0.3978	0.4000
10	0.3368	0.3702	0.3829	0.3896	0.3937	0.3965	0.3985	0.4000
11	0.3538	0.3788	0.3879	0.3927	0.3956	0.3975	0.3989	0.4000
12	0.3666	0.3850	0.3915	0.3949	0.3969	0.3983	0.3993	0.4000
13	0.3762	0.3894	0.3941	0.3964	0.3978	0.3988	0.3995	0.4000
14	0.3831	0.3926	0.3958	0.3975	0.3985	0.3992	0.3996	0.4000
15	0.3881	0.3948	0.3971	0.3983	0.3989	0.3994	0.3997	0.4000
16	0.3916	0.3964	0.3980	0.3988	0.3993	0.3996	0.3998	0.4000
17	0.3941	0.3975	0.3986	0.3992	0.3995	0.3997	0.3999	0.4000
18	0.3959	0.3982	0.3990	0.3994	0.3996	0.3998	0.3999	0.4000
19	0.3971	0.3988	0.3993	0.3996	0.3998	0.3999	0.3999	0.4000
20	0.3980	0.3980	0.3991	0.3995	0.3997	0.3998	0.3999	0.4000
21	0.3986	0.3986	0.3994	0.3997	0.3998	0.3999	0.3999	0.4000
22	0.3990	0.3996	0.3998	0.3999	0.3999	0.4000	0.4000	0.4000
23	0.3993	0.3997	0.3998	0.3999	0.3999	0.4000	0.4000	0.4000
24	0.3995	0.3998	0.3999	0.3999	0.4000	0.4000	0.4000	0.4000
25	0.3997	0.3999	0.3999	0.4000	0.4000	0.4000	0.4000	0.4000
26	0.3998	0.3999	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000
27	0.3998	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000
28	0.3999	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000
29	0.3999	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000
30	0.3999	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000

Panel B.	ROE = 40% λ=0.60							
Time (*t*)	$g_0 = 5\%$	$g_0 = 10\%$	$g_0 = 15\%$	$g_0 = 20\%$	$g_0 = 25\%$	$g_0 = 30\%$	$g_0 = 35\%$	$g_0 = 40\%$
1	0.0580	0.1134	0.1664	0.2171	0.2657	0.3123	0.3570	0.4000
2	0.0670	0.1278	0.1832	0.2340	0.2805	0.3235	0.3632	0.4000
3	0.0771	0.1432	0.2003	0.2503	0.2944	0.3335	0.3685	0.4000
4	0.0884	0.1593	0.2174	0.2660	0.3072	0.3425	0.3731	0.4000
5	0.1007	0.1760	0.2343	0.2808	0.3188	0.3504	0.3771	0.4000
6	0.1142	0.1930	0.2506	0.2947	0.3294	0.3574	0.3806	0.4000
7	0.1287	0.2101	0.2663	0.3074	0.3388	0.3635	0.3835	0.4000
8	0.1441	0.2271	0.2811	0.3190	0.3471	0.3688	0.3860	0.4000
9	0.1602	0.2437	0.2949	0.3296	0.3545	0.3734	0.3881	0.4000
10	0.1769	0.2597	0.3077	0.3390	0.3610	0.3773	0.3900	0.4000
11	0.1940	0.2749	0.3193	0.3473	0.3666	0.3807	0.3915	0.4000
12	0.2111	0.2891	0.3298	0.3547	0.3715	0.3837	0.3928	0.4000
13	0.2281	0.3023	0.3391	0.3611	0.3757	0.3861	0.3939	0.4000
14	0.2446	0.3144	0.3475	0.3667	0.3794	0.3883	0.3949	0.4000
15	0.2606	0.3254	0.3548	0.3716	0.3825	0.3901	0.3957	0.4000
16	0.2757	0.3352	0.3612	0.3758	0.3851	0.3916	0.3964	0.4000
17	0.2899	0.3440	0.3668	0.3794	0.3874	0.3929	0.3969	0.4000
18	0.3031	0.3518	0.3717	0.3825	0.3893	0.3940	0.3974	0.4000
19	0.3151	0.3586	0.3759	0.3852	0.3910	0.3949	0.3978	0.4000
20	0.3260	0.3645	0.3795	0.3874	0.3924	0.3957	0.3982	0.4000
21	0.3358	0.3697	0.3826	0.3894	0.3935	0.3964	0.3984	0.4000
22	0.3445	0.3742	0.3852	0.3910	0.3946	0.3970	0.3987	0.4000
23	0.3522	0.3780	0.3875	0.3924	0.3954	0.3974	0.3989	0.4000
24	0.3589	0.3813	0.3894	0.3936	0.3961	0.3978	0.3991	0.4000
25	0.3648	0.3841	0.3910	0.3946	0.3967	0.3982	0.3992	0.4000
26	0.3700	0.3866	0.3924	0..954	0.3972	0.3985	0.3993	0.4000
27	0.3744	0.3886	0.3936	0.3961	0.3977	0.3987	0.3994	0.4000
28	0.3782	0.3904	0.3946	0.3967	0.3980	0.3989	0.3995	0.4000
29	0.3815	0.3918	0.3954	0.3972	0.3983	0.3991	0.3996	0.4000
30	0.3843	0.3931	0.3961	0.3977	0.3986	0.3992	0.3997	0.4000

Panel C.	**ROE = 50%** λ **=0.95**							
Time (*t*)	$g_0 = 5\%$	$g_0 = 10\%$	$g_0 = 15\%$	$g_0 = 20\%$	$g_0 = 25\%$	$g_0 = 30\%$	$g_0 = 35\%$	$g_0 = 40\%$
1	0.0743	0.1411	0.2013	0.2559	0.3056	0.3511	0.3929	0.4314
2	0.1077	0.1909	0.2572	0.3111	0.3560	0.3938	0.4261	0.4541
3	0.1508	0.2464	0.3124	0.3607	0.3796	0.4268	0.4503	0.4698
4	0.2021	0.3021	0.3618	0.4014	0.4297	0.4508	0.4672	0.4803
5	0.2581	0.3530	0.4022	0.4324	0.4528	0.4675	0.4786	0.4873
6	0.3132	0.3953	0.4331	0.4548	0.4689	0.4789	0.4862	0.4919
7	0.3625	0.4279	0.4552	0.4703	0.4798	0.4863	0.4911	0.4948
8	0.4028	0.4516	0.4706	0.4807	0.4869	0.4912	0.4943	0.4967
9	0.4335	0.4681	0.4809	0.4875	0.4916	0.4944	0.4964	0.4979
10	0.4555	0.4792	0.4877	0.4920	0.4946	0.4964	0.4977	0.4986
11	0.4708	0.4866	0.4921	0.4949	0.4966	0.4977	0.4985	0.4991
12	0.4810	0.4914	0.4949	0.4967	0.4978	0.4985	0.4991	0.4995
13	0.4877	0.4945	0.4968	0.4979	0.4986	0.4991	0.4994	0.4997
14	0.4921	0.4965	0.4979	0.4987	0.4991	0.4994	0.4996	0.4998
15	0.4950	0.4978	0.4987	0.4992	0.4994	0.4996	0.4998	0.4999
16	0.4968	0.4986	0.4992	0.4995	0.4996	0.4998	0.4998	0.4999
17	0.4980	0.4991	0.4995	0.4997	0.4998	0.4998	0.4999	0.4999
18	0.4987	0.4994	0.4997	0.4998	0.4999	0.4999	0.4999	0.5000
19	0.4992	0.4996	0.4998	0.4999	0.4999	0.4999	0.5000	0.5000
20	0.4995	0.4998	0.4999	0.4999	0.4999	0.5000	0.5000	0.5000
21	0.4997	0.4999	0.4999	0.4999	0.5000	0.5000	0.5000	0.5000
22	0.4998	0.4999	0.4999	0.5000	0.5000	0.5000	0.5000	0.5000
23	0.4999	0.4999	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
24	0.4999	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
25	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
26	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
27	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
28	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
29	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
30	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

Panel D.	ROE = 50% λ =0.60							
Time (*t*)	$g_0 = 5\%$	$g_0 = 10\%$	$g_0 = 15\%$	$g_0 = 20\%$	$g_0 = 25\%$	$g_0 = 30\%$	$g_0 = 35\%$	$g_0 = 40\%$
1	0.0605	0.1182	0.1734	0.2262	0.2767	0.3251	0.3715	0.4160
2	0.0729	0.1387	0.1984	0.2529	0.3028	0.3486	0.3909	0.4300
3	0.0872	0.1611	0.2245	0.2795	0.3277	0.3702	0.4081	0.4419
4	0.1037	0.1854	0.2512	0.3055	0.3510	0.3897	0.4231	0.4520
5	0.1225	0.2110	0.2779	0.3303	0.3724	0.4071	0.4360	0.4606
6	0.1433	0.2374	0.3039	0.3534	0.3917	0.4222	0.4470	0.4677
7	0.1662	0.2642	0.3288	0.3746	0.4088	0.4353	0.4564	0.4736
8	0.1908	0.2906	0.3521	0.3937	0.4237	0.4464	0.4642	0.4785
9	0.2166	0.3162	0.3734	0.4105	0.4365	0.4558	0.4707	0.4825
10	0.2432	0.3403	0.3925	0.4252	0.4475	0.4637	0.4761	0.4858
11	0.2699	0.3626	0.4095	0.4378	0.4568	0.4703	0.4805	0.4884
12	0.2962	0.3829	0.4243	0.4486	0.4645	0.4758	0.4841	0.4906
13	0.3215	0.4010	0.4371	0.4577	0.4709	0.4803	0.4871	0.4924
14	0.3453	0.4170	0.4480	0.4653	0.4763	0.4839	0.4896	0.4939
15	0.3672	0.4308	0.4571	0.4716	0.4807	0.4870	0.4915	0.4950
16	0.3870	0.4426	0.4648	0.4768	0.4843	0.4894	0.4931	0.4960
17	0.4047	0.4526	0.4712	0.4811	0.4872	0.4914	0.4945	0.4967
18	0.4201	0.4201	0.4610	0.4765	0.4846	0.4897	0.4931	0.4955
19	0.4335	0.4681	0.4809	0.4875	0.4916	0.4944	0.4964	0.4979
20	0.4449	0.4739	0.4844	0.4899	0.4932	0.4955	0.4971	0.4983
21	0.4546	0.4787	0.4874	0.4918	0.4945	0.4963	0.4976	0.4986
22	0.4627	0.4827	0.4898	0.4934	0.4956	0.4970	0.4981	0.4989
23	0.4694	0.4859	0.4917	0.4946	0.4964	0.4976	0.4985	0.4991
24	0.4750	0.4886	0.4933	0.4957	0.4971	0.4981	0.4988	0.4993
25	0.4796	0.4907	0.4946	0.4965	0.4977	0.4984	0.4990	0.4994
26	0.4834	0.4925	0.4956	0.4972	0.4981	0.4987	0.4992	0.4995
27	0.4866	0.4939	0.4964	0.4977	0.4985	0.4990	0.4993	0.4996
28	0.4891	0.4951	0.4971	0.4981	0.4988	0.4992	0.4995	0.4997
29	0.4912	0.960	0.4977	0.4985	0.4990	0.4993	0.4996	0.4998
30	0.4928	0.4968	0.4981	0.4988	0.4992	0.4995	0.4997	0.4998

Figure 1. Sensitivity Analysis of the Optimal Growth Rate

The figures show the sensitivity analysis that how (i) the time horizon (t), (ii) the degree of market perfection (λ), (iii) the rate of return on equity $\left(\frac{\bar{r}}{1-L}\right)$, and (iv) the initial growth rate (g_o) affect the optimal growth rate. Figure 1.a presents the mean-reverting process assuming that the rate of return on equity is 40 percent and the degree of market perfection is 0.95. Each line shows the optimal growth rates over 30 years with different initial growth rates. Figure 1.b presents the mean-reverting process assuming that the rate of return on equity is 40 percent and the degree of market perfection is 0.60. Figure 1.c presents the mean-reverting process of the optimal growth rate assuming that the rate of return on equity is 50 percent and the degree of market perfection is 0.95. Figure 1.d presents the mean-reverting process of the optimal growth rate assuming that the rate of return on equity is 50 percent and the degree of market perfection is 0.60

Figure 1.a $ROE = 40\%;\ \lambda = 0.95$

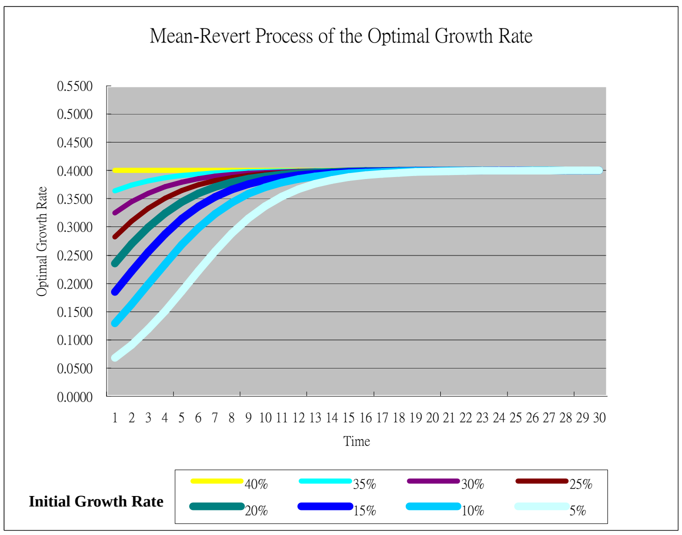

Figure 1.b $ROE = 40\%;\ \lambda = 0.60$

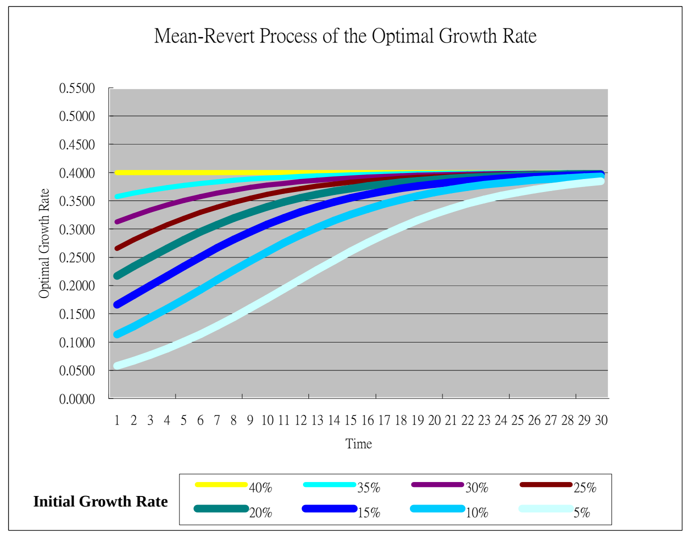

Figure 1.c $ROE = 50\%;\ \lambda = 0.95$

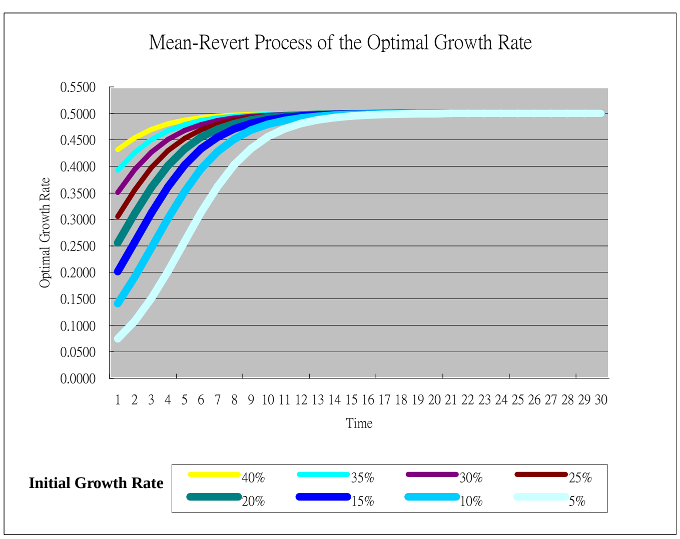

Figure 1.d $ROE = 50\%;\ \lambda = 0.60$

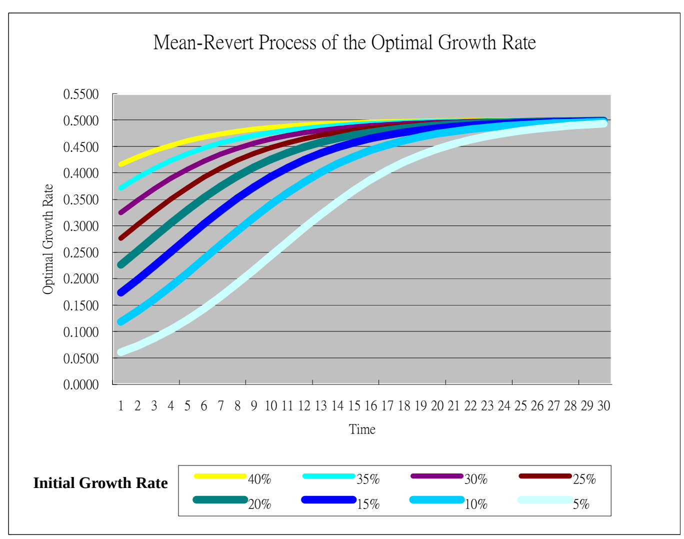

Figure 2. Optimal Growth Rate with respect to the Rate of Return on Equity and the Degree of Market Perfection

The figure shows the optimal growth rates using different rates of return on equity and degrees of market perfection when the initial growth rate is 5 percent and the time is 3. The rates of return on equity range from 5 percent to 50 percent. The degrees of market perfection range from 0.05 to 1.