Alternative Methods to Estimate Implied Variance: Review and Comparison

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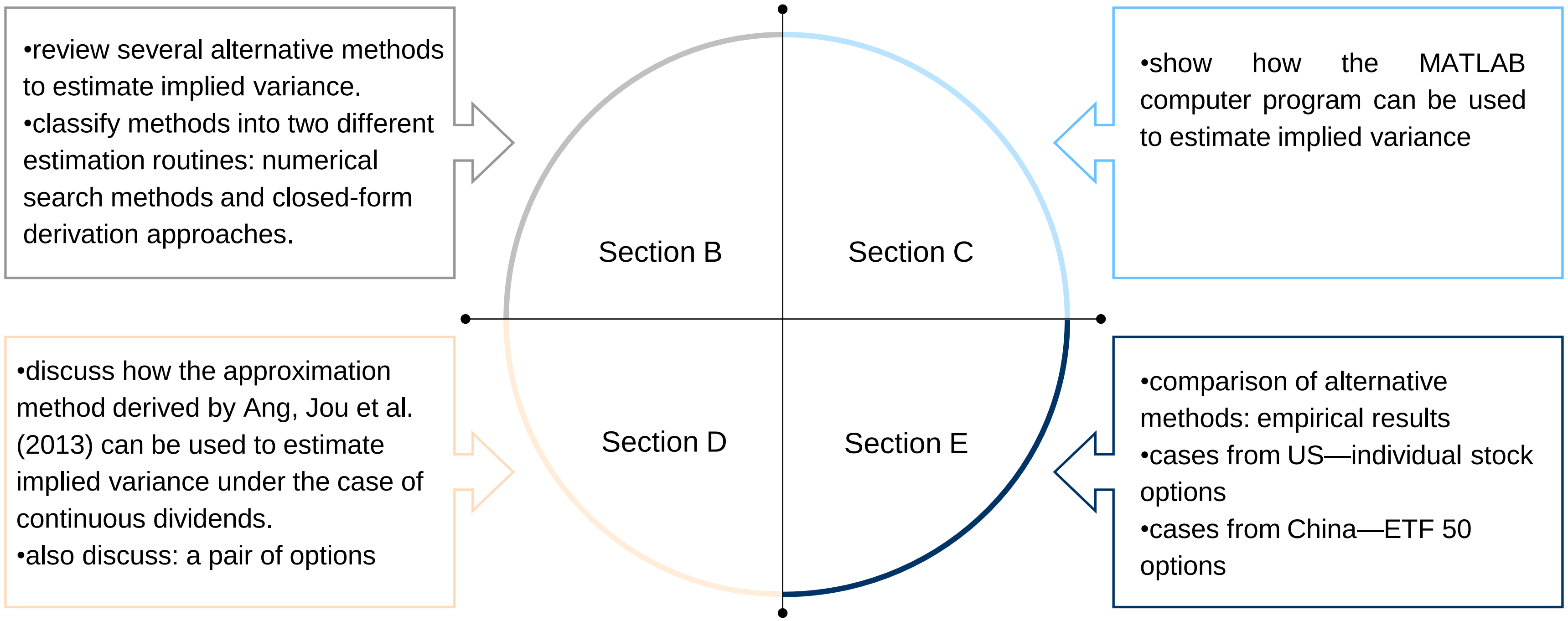
Abstract

- The main purpose of this paper is to review and compare alternative methods for estimating implied variance. In this paper, we first review several alternative methods to estimate implied variance. Then we show how the MATLAB computer program can be used to estimate implied variance based upon the Black-Scholes model. In addition, we also discuss how the approximation method derived by Ang, Jou et al. (2013) can be used to estimate implied variance and implied stock price per share. Real world data from US individual stock options are used to compare the estimation results using three typical alternative methods: regression method proposed by Lai, Lee et al, MATLAB computer program approach and approximation method derived by Ang, Jou et al. Also, this paper presents the empirical results of China ETF 50 options which were new in the financial markets.

Introduction

- It is well known that implied variance estimation is important for evaluating option pricing. In this paper, we first review several alternative methods to estimate implied variance in Section B. We classify them into two different estimation routines: numerical search methods and closed-form derivation approaches. Closed-form derivation approaches took use of either Taylor expansion or inverse function to calculate the analytical solutions for the ISD.
- In Section C, we show how the MATLAB computer program can be used to estimate implied variance. This computer program is based upon the Black-Scholes model using Newton-Raphson method.
- In Section D, we discuss how the approximation method derived by Ang, Jou et al. (2013) can be used to estimate implied variance under the case of continuous dividends. This approximation method can also estimate implied volatility from two options with the same maturity, but different exercise prices and values.
- In Section E, real data from American option markets are used to compare the performances of three typical alternative methods: regression method proposed by Lai, Lee et al, MATLAB computer program approach and approximation method derived by Ang, Jou et al. The results are presented in Section E. Also, this paper presents the empirical results of China ETF 50 options which were new in the financial markets. Section F summarizes the paper.

Framework Summary



Alternative methods to estimate implied variance

Numerical Search	Closed-form Derivation
Trial and error **Latane and Rendleman (1976)**	Taylor Series Expansion
	First-order expansion: Brenner and Subrahmanyam (1988); Corrado and Miller (1996) Second-order expansion: Chance (1996) Third-order expansion: Li (2005)
Choose an initial point, iterative algorithm **Manaster and Koehler (1982)**	Inverse Function
	Estimate parameters by regression: Lai, Lee et al. (1992)

Numerical search method **Latane and Rendleman (1976)**

$$C = SN(d_1) - Xe^{-rT}N(d_2)$$

Within ± 0.001 of the observed actual call price

$$d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Trail and error

$$d_2 = d_1 - \sigma\sqrt{T}$$

S=current market price of the underlying stock;
X=exercise price;
r=continuous constant interest rate;
T=remaining life of the option

Numerical search method

Manaster and Koehler (1982), choose an initial point

$$C = f(S, X, r, T, \sigma) = f(\sigma)$$

strictly monotone increasing

$$\lim_{\sigma \to 0^+} f(\sigma) = \max(0, S - Xe^{-rT})$$

$$\lim_{\sigma \to \infty} f(\sigma) = S$$

$$\max(0, S - Xe^{-rT}) < C < S$$

Ensure: a positive solution of implied standard deviation σ^*

Mean-Value Theorem. *Let f be a continuous function on the closed interval $[a,b]$, and can be differentiable on the open interval (a,b), where $a < b$. There exists some $c \in (a,b)$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{|\sigma_{n+1} - \sigma^*|}{|\sigma_n - \sigma^*|} = \left|1 - \frac{f(\sigma_n) - C}{f'(\sigma_n)(\sigma_n - \sigma^*)}\right| = \left|1 - \frac{f'(\lambda\sigma^* + (1-\lambda)\sigma_n)}{f'(\sigma_n)}\right|$$

maximize

initial point

$$\sigma_1^2 = \left|\ln\left(\frac{S}{X}\right) + rT\right|\frac{2}{T}$$

Closed-Form Derivation: Taylor Series Expansion

- First-Order Taylor Series Expansion: Brenner and Subrahmanyam (1988)

$$S = Xe^{-rT}$$

At-the-money

$$N(d_1) = N(0) + N'(0)d_1 + \cdots = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}d_1 + o(d_1)$$

$$N(d_1) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}d_1 = \frac{1}{2} + \frac{1}{2\sqrt{2\pi}}\sigma\sqrt{T}$$

$$N(d_2) \approx 1 - N(d_1) = \frac{1}{2} - \frac{1}{2\sqrt{2\pi}}\sigma\sqrt{T}$$

$$C = \frac{S\sigma\sqrt{T}}{\sqrt{2\pi}} \qquad \sigma = \frac{C\sqrt{2\pi}}{S\sqrt{T}}$$

Limitation

Note that Brenner and Subramanyam's method can only be used to estimate implied standard deviation from at-the-money or at least not too far in- or out-of-the-money options.

Closed-Form Derivation: Taylor Series Expansion

- First-Order Taylor Series Expansion: Brenner and Subrahmanyam (1988)

$$N(z) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}(z - \frac{z^3}{6} + \cdots)$$

$$C = S(\frac{1}{2} + \frac{d_1}{\sqrt{2\pi}}) - Xe^{-rT}(\frac{1}{2} + \frac{d_1 - \sigma\sqrt{T}}{\sqrt{2\pi}})$$

$$d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$C = S(\frac{1}{2} + \frac{\ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{2\pi T}}) - K(\frac{1}{2} + \frac{\ln(S/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{2\pi T}})$$

$$\sigma^2 T(S + K) - \sigma\sqrt{T}[(2\sqrt{2\pi}C - \sqrt{2\pi}(S - K)] + 2(S - K)\ln(S/K) = 0$$

Closed-Form Derivation: Taylor Series Expansion

- Second-Order Taylor Series Expansion: Chance (1996)

$$\sigma^* = \frac{C^*\sqrt{2\pi}}{S\sqrt{T}}$$

At-the-money call

Brenner and Subrahmanyam's ISD

$$\Delta C^* = C - C^* \quad \begin{cases} \Delta X^* = X - X^* \\ \Delta\sigma^* = \sigma - \sigma^* \end{cases}$$

$$\Delta C^* = \frac{\partial C^*}{\partial X^*}(\Delta X^*) + \frac{1}{2}\frac{\partial^2 C^*}{\partial X^{*2}}(\Delta X^*)^2 + \frac{\partial C^*}{\partial \sigma^*}(\Delta\sigma^*) + \frac{1}{2}\frac{\partial^2 C^*}{\partial \sigma^{*2}}(\Delta\sigma^*)^2 + \frac{\partial^2 C^*}{\partial\sigma^*\partial X^*}(\Delta\sigma^*\Delta X^*)$$

$$a(\Delta\sigma^*)^2 + b(\Delta\sigma^*) + c = 0$$

$$a = \frac{1}{2}\frac{\partial^2 C^*}{\partial\sigma^{*2}} \quad b = \frac{\partial C^*}{\partial\sigma^*} + \frac{\partial^2 C^*}{\partial\sigma^*\partial X^*}(\Delta X^*) \quad c = C^* - C + \frac{\partial C^*}{\partial X^*}(\Delta X^*) + \frac{1}{2}\frac{\partial^2 C^*}{\partial X^{*2}}(\Delta X^*)^2$$

Closed-Form Derivation: Taylor Series Expansion

- Third-Order Taylor Series Expansion: Li (2005)

$$C = S(\frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} - \frac{{d_1}^3}{6\sqrt{2\pi}}) - Xe^{-rT}(\frac{1}{2} + \frac{d_2}{\sqrt{2\pi}} - \frac{{d_2}^3}{6\sqrt{2\pi}})$$

$$\sigma = \frac{2\sqrt{2}}{\sqrt{T}} z - \frac{1}{\sqrt{T}} \sqrt{8z^2 - \frac{6\alpha}{\sqrt{2}z}}$$

Where $\alpha = \frac{\sqrt{2\pi}C}{S}$

$$z = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3\alpha}{\sqrt{32}}\right)\right]$$

Closed-Form Derivation: Regression Method

Lai, Lee et al. (1992)

$$\frac{\partial C}{\partial S} = N(d_1)$$

$$\frac{\partial C}{\partial X} = -e^{-rT} N(d_2)$$

$$d_1 = N^{-1}[(\frac{\partial C}{\partial S})]$$

$$d_2 = d_1 - \sigma\sqrt{T} = N^{-1}[e^{rT}(-\frac{\partial C}{\partial X})]$$

$$\sigma = \{[N^{-1}(\frac{\partial C}{\partial S}) - N^{-1}[e^{rT}(-\frac{\partial C}{\partial X})]\} \Big/ \sqrt{T}$$

$$C = (\frac{\partial C}{\partial S})S + (\frac{\partial C}{\partial X})X = \beta_S S + \beta_X X$$

Regression

$$C_{it} = \alpha + \beta_S S_t + \beta'_X e^{-rT} X_{it} + \varepsilon_{it}$$

$$\sigma = [N^{-1}(\hat{\beta}_S) - N^{-1}(-\hat{\beta}'_X)] \Big/ \sqrt{T}$$

This alternative approach would work best for index options, where there are many simultaneous quotes.

MATLAB approach to estimate implied variance

$$C^M_{j,t} - C^T_{j,t}(\sigma_0) = \left[\frac{\partial C^T_{j,t}}{\partial \sigma} \bigg|_{\sigma_0} \right] (\sigma - \sigma_0) + e_{j,t}$$

$$\frac{\partial C^F_{t,j}}{\partial \sigma} = Xe^{-r\tau}\sqrt{\tau}N'(d_1) = Xe^{-r\tau}\frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-d_1^2/2}$$

$$\left| \frac{\sigma_1 - \sigma_0}{\sigma_0} \right| < .001$$

Tolerance level

Inputs:

Price - Current price of the underlying asset.
Strike - Strike (i.e., exercise) price of the option.
Rate - Annualized continuously compounded risk-free rate of return over the life of the option, expressed as a positive decimal number.
Time - Time to expiration of the option, expressed in years.
Value - Price (i.e., value) of a European option from which the implied volatility of the underlying asset is derived.

Output:

Volatility - Implied volatility of the underlying asset derived from European option prices, expressed as a decimal number. If no solution can be found, a NaN (i.e., Not-a-Number) is returned.

Examples

Volatility = blsimpv(Price, Strike, Rate, Time, Value, Limit, Yield, Tolerance, Class)
Volatility = blsimpv(90, 95, 0.03, 0.25, 5,[],0.05,[], {'Call'})
Volatility = blsimpv(90, 95, 0.03, 0.25, 5,[],0.05,[], true)

Approximation approach to estimate implied variance

Ang, Jou et al. (2009)

$$C = S'N(d_1) - KN(d_2)$$

$$d_1 = \frac{\ln(S/X) + (r + \frac{\sigma^2}{2} - q)T}{\sigma\sqrt{T}} = \frac{\ln(S'/K)}{\sigma\sqrt{T}} + \sigma\sqrt{T}/2$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

BS model with dividends

- Let $L' = \ln(S'/K)/\sigma\sqrt{T}$

$$N(L' + \sigma\sqrt{T}/2)$$
$$= N(L') + N'(L')\sigma\sqrt{T}/2 + N''(L')(\sigma\sqrt{T}/2)^2/2 + e_1$$
$$= N(L') + N'(L')(\sigma\sqrt{T}/2)[1 - \ln(S'/K)/4] + e_1$$

$$N(L' - \sigma\sqrt{T}/2)$$
$$= N(L') - N'(L')\sigma\sqrt{T}/2 + N''(L')(\sigma\sqrt{T}/2)^2/2 + e_2$$
$$= N(L') - N'(L')(\sigma\sqrt{T}/2)[1 + \ln(S'/K)/4] + e_2$$

$$\sigma^2 T[8(S' + K) - 2(S' - K)\ln(S'/K)] - 8\sigma\sqrt{T}\sqrt{2\pi}(2C - S' + K)$$
$$+ \ln(S'/K)[(S' - K)(16 + (\ln(S'/K))^2) - 4(S' + K)\ln(S'/K)] = 0$$

Approximation approach to estimate implied variance

Ang, Jou et al. (2009)

$$\sigma\sqrt{T} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Call option case

Where $a = 8(S' + K) - 2(S' - K)\ln(S'/K)$

$$b = -8\sqrt{2\pi}(2C - S' + K)$$

$$c = \ln(S'/K)[(S' - K)(16 + \ln(S'/K))^2] - 4(S' + K)\ln(S'/K)]$$

$$\sigma\sqrt{T} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Put option case

Where $a = 8(S' + K) - 2(S' - K)\ln(S'/K)$

$$b = -8\sqrt{2\pi}(2P - K + S')$$

$$c = \ln(S'/K)[(S' - K)(16 + \ln(S'/K)^2)] - 4(S' + K)\ln(S'/K)]$$

Approximation approach to estimate implied variance

A pair of two call options

$$C_1 = C_2 - N(\ln(S'/K_2)/\sigma\sqrt{T} - \sigma\sqrt{T}/2)(K_1 - K_2) + \varepsilon_1$$

$$C_2 = C_1 - N(\ln(S'/K_1)/\sigma\sqrt{T} - \sigma\sqrt{T}/2)(K_2 - K_1) + \varepsilon_2$$

$$N^{-1}[(C_1 - C_2)/(K_2 - K_1)] = \ln(S'/K_1)/\sigma\sqrt{T} - \sigma\sqrt{T}/2 + \eta_1$$

$$N^{-1}[(C_1 - C_2)/(K_2 - K_1)] = \ln(S'/K_2)/\sigma\sqrt{T} - \sigma\sqrt{T}/2 + \eta_2$$

$$\sigma^2 T + 2N^{-1}[(C_1 - C_2)/(K_2 - K_1)](\sigma\sqrt{T}) - \ln(S'/K_1) - \ln(S'/K_2) = 0$$

Solution

$$\sigma\sqrt{T} = -N^{-1}((C_1 - C_2)/(K_2 - K_1)) \pm \sqrt{[N^{-1}((C_1 - C_2)/(K_2 - K_1))]^2 + \ln(S'^2/K_1K_2)}$$

Approximation approach to estimate implied variance

A pair of two put options

$$P_1 = P_2 + (K_1 - K_2) - N(\ln(S'/K_2)/\sigma\sqrt{T} - \sigma\sqrt{T}/2)(K_1 - K_2) + \delta_1$$

$$P_2 = P_1 + (K_2 - K_1) - N(\ln(S'/K_2)/\sigma\sqrt{T} - \sigma\sqrt{T}/2)(K_2 - K_1) + \delta_2$$

$$N^{-1}((P_1 - P_2)/(K_2 - K_1) + 1) = \ln(S'/K_2)/\sigma\sqrt{T} - \sigma\sqrt{T}/2 + \gamma_1$$

$$N^{-1}((P_1 - P_2)/(K_2 - K_1) + 1) = \ln(S'/K_1)/\sigma\sqrt{T} - \sigma\sqrt{T}/2 + \gamma_2$$

$$\sigma^2 T + 2N^{-1}[(P_1 - P_2)/(K_2 - K_1) + 1](\sigma\sqrt{T}) - \ln(S'/K_1) - \ln(S'/K_2) = 0$$

Solution

$$\sigma\sqrt{T} = -N^{-1}((P_1 - P_2)/(K_2 - K_1) + 1) \pm \sqrt{[N^{-1}((P_1 - P_2)/(K_2 - K_1) + 1)]^2 + \ln(S'^2/K_1K_2)}$$

Some empirical results

- Cases from US—Individual Stock Options

Security ID	Ticker	Company Name	SIC Code	Industry
101594	AAPL	Apple Inc.	3571	Electronic Computers
104533	XOM	Exxon Mobil Corporation	2911	Petroleum Refining
121812	GOOGL	Google Inc.	7375	Information Retrieval Services
107525	MSFT	Microsoft Corporation	7372	Prepackaged Software
106566	JNJ	Johnson & Johnson	2834	Pharmaceutical Preparations
111953	WFC	Wells Fargo & Company	6022	State Commercial Banks
105169	GE	General Electric Company	3511	Steam, Gas, and Hydraulic Turbines, and Turbine Engine
111860	WMT	Wal-Mart Stores Inc.	5331	Variety Stores
102968	CVX	Chevron Corporation	2911	Petroleum Refining
109224	PG	The Procter & Gamble Company	2841	Soap and Other Detergents
102936	JPM	JPMorgan Chase & Co.	6211	Security Brokers, Dealers & Flotation Companies
111668	VZ	Verizon Communications Inc.	4812	Radiotelephone Communications
108948	PFE	Pfizer Inc.	2834	Pharmaceutical Preparations
106276	IBM	International Business Machines Corporation	3571	Electronic Computers
109775	T	AT&T, Inc.	4812	Radiotelephone Communications

relative large market values

Cases from US—Individual Stock Options

Ticker	IV-matlab	IV-approximation	IV-regression
AAPL	0.387	0.332	0.458
XOM	0.208	0.185	0.106
GOOGL	0.376	0.389	0.305
MSFT	0.327	0.341	0.298
JNJ	0.223	0.216	No positive solution
WFC	0.312	0.301	No positive solution
GE	0.176	0.142	0.224
WMT	0.127	0.124	0.164
CVX	0.306	0.285	0.367
PG	0.209	0.185	No positive solution
JPM	0.189	0.192	0.135
VZ	0.169	0.174	0.247
PFE	0.216	0.208	0.185
IBM	0.463	0.457	No positive solution
T	0.186	0.189	0.264

Some empirical results

- Cases from China—ETF 50 Options

Option Ticker	Exercise Price	Expiration Date	IV-matlab	IV-approximation
10000021.SH	2.20	2015-06-25	0.515	0.504
10000022.SH	2.25	2015-06-25	0.486	0.498
10000023.SH	2.30	2015-06-25	0.417	0.423
10000024.SH	2.35	2015-06-25	0.439	0.424
10000025.SH	2.40	2015-06-25	0.489	0.478
10000031.SH	2.20	2015-09-24	0.426	0.442
10000032.SH	2.25	2015-09-24	0.435	0.447
10000033.SH	2.30	2015-09-24	0.417	0.428
10000034.SH	2.35	2015-09-24	0.422	0.432
10000035.SH	2.40	2015-09-24	0.443	0.454
10000045.SH	2.45	2015-06-25	0.428	0.436
10000047.SH	2.45	2015-09-24	0.393	0.408
10000053.SH	2.50	2015-06-25	0.443	0.428
10000055.SH	2.50	2015-09-24	0.415	0.420
10000061.SH	2.55	2015-06-25	0.442	0.438
10000063.SH	2.55	2015-09-24	0.398	0.416
10000069.SH	2.60	2015-06-25	0.420	0.431
10000071.SH	2.60	2015-09-24	0.409	0.412
10000077.SH	2.65	2015-06-25	0.426	0.419
10000079.SH	2.65	2015-09-24	0.416	0.428
10000085.SH	2.70	2015-06-25	0.427	0.434
10000087.SH	2.70	2015-09-24	0.414	0.419
10000093.SH	2.75	2015-06-25	0.417	0.426
10000095.SH	2.75	2015-09-24	0.405	0.411
10000101.SH	2.80	2015-06-25	0.427	0.443
10000103.SH	2.80	2015-09-24	0.409	0.401
10000123.SH	2.85	2015-06-25	0.441	0.432
10000125.SH	2.85	2015-09-24	0.417	0.422

In Chinese financial market, there were no stock options in the exchange until February, 2015. Now, the only traded options in China are ETF 50 options.

THANKS!
Q&A