Cross-Sectional Asset Pricing Puzzles: An Equilibrium Perspective†

Doron Avramov, Scott Cederburg, and Satadru Hore∗

ABSTRACT

This paper proposes an intertemporal asset pricing model that resolves the negative cross-sectional relations between expected stock return and dispersion, idiosyncratic volatility (IV), and credit risk. All three puzzling effects naturally emerge in the cross section of an economy characterized by recursive preferences and persistent dividend and consumption growth rates. The equilibrium cross section of expected return is driven by time-varying exposure to an economic growth factor. The three effects emerge through the interaction of firm cash flow timing and investor aversion to shocks in economic growth. Specifically, low expected growth firms derive their values primarily from short-run cash flows. Such firms exhibit high dispersion, IV, and credit risk due to their high price sensitivity to idiosyncratic shocks. However, they have low exposure to economic growth shocks thereby exhibiting low growth beta and expected return. In contrast, firms with values weighted towards long-run cash flows have greater exposure to aggregate risk and are relatively insensitive to idiosyncratic cash flow shocks. Thus they are characterized by high expected return coupled with low dispersion, IV, and credit risk levels.

†We thank David Bates, George Skoulakis, Weina Zhang, seminar participants at the Hebrew University of Jerusalem, and participants at the Third Singapore International Conference on Finance for helpful comments. All errors are our own.

∗Doron Avramov (davramov@rhsmith.umd.edu) is from the University of Maryland and the Hebrew University of Jerusalem, Scott Cederburg (scott-cederburg@uiowa.edu) is from the University of Iowa, and Satadru Hore (satadru.hore@bos.frb.org) is from the Federal Reserve Bank of Boston.
Cross-Sectional Asset Pricing Puzzles:  
An Equilibrium Perspective  

ABSTRACT  
This paper proposes an intertemporal asset pricing model that resolves the negative cross-sectional relations between expected stock return and dispersion, idiosyncratic volatility (IV), and credit risk. All three puzzling effects naturally emerge in the cross section of an economy characterized by recursive preferences and persistent dividend and consumption growth rates. The equilibrium cross section of expected return is driven by time-varying exposure to an economic growth factor. The three effects emerge through the interaction of firm cash flow timing and investor aversion to shocks in economic growth. Specifically, low expected growth firms derive their values primarily from short-run cash flows. Such firms exhibit high dispersion, IV, and credit risk due to their high price sensitivity to idiosyncratic shocks. However, they have low exposure to economic growth shocks thereby exhibiting low growth beta and expected return. In contrast, firms with values weighted towards long-run cash flows have greater exposure to aggregate risk and are relatively insensitive to idiosyncratic cash flow shocks. Thus they are characterized by high expected return coupled with low dispersion, IV, and credit risk levels.
Introduction

This paper derives an intertemporal asset pricing model which offers a unified resolution for the puzzling negative relations between expected stock return and analysts’ earnings forecast dispersion, idiosyncratic volatility (IV), and credit risk. Our paradigm incorporates recursive preferences and persistent consumption and dividend growth rates, giving rise to the long-run risk economy originated by Bansal and Yaron (2004). Furthermore, our model is based on a natural cross section of firms, where firm-level dividends add up to the aggregate dividend. Cross-sectionally, firms differ through loadings on an economic growth risk factor. In particular, factor loadings vary with expected cash flow timing, summarized by a characteristic labeled relative share – a term coined by Menzly, Santos, and Veronesi (2004). Relative share is the long-run expected dividend share of a firm as a proportion of its current dividend share, with dividend share being the fraction of the dividend paid by the firm relative to the aggregate dividend. In the time series, the economy-wide risk premium varies as aggregate exposure to long-run risk changes with shocks to economic growth, and the firm factor loading varies with the dynamics of firm relative share and the aggregate economic state.

The three cross-sectional puzzles analyzed in this paper have been documented in recent work. Specifically, Dichev (1998), Avramov, Chordia, Jostova, and Philipov (2008), and Campbell, Hilscher, and Szilagyi (2008) show that financially distressed stocks deliver abnormally low returns and moreover higher rated stocks considerably outperform their lower rated counterparts. Diether, Malloy, and Scherbina (2002) (hereafter DMS) demonstrate that firms with more uncertain earnings (higher forecast dispersion) underperform relative to otherwise similar stocks. Ang, Hodrick, Xing, and Zhang (2006, 2009) (hereafter AHXZ)
document a negative relation between IV and abnormal returns in US and international markets. The dispersion, IV, and credit risk effects are apparently anomalous because, while investors are expected to discount uncertainty about firm fundamentals, they ultimately appear to pay a premium for bearing such uncertainty. The IV effect is also at odds with the theories of Merton (1987) and Barberis and Huang (2002) as well as with common sense that investors should be compensated for their inability to properly diversify their portfolios. All three puzzling effects remain unexplained by the CAPM of Sharpe (1964) and Lintner (1965) and the CAPM augmented by the size and value spreads of Fama and French (1993) as well as the momentum portfolio of Jegadeesh and Titman (1993).

Economic theory has attempted to rationalize the dispersion, IV, and credit risk effects. Johnson (2004) interprets dispersion as a proxy for unpriced information risk arising from unobservable asset value. Based on Merton (1974), Johnson shows that in the presence of leverage, firm value rises (and expected return falls) as unpriced volatility risk rises. Garlappi, Shu, and Yan (2008) attribute the negative relation between distress risk and return to shareholders’ ability to extract rents from debt obligation renegotiation upon experiencing financial distress. Barinov (2007) offers an explanation for the IV effect in a real option framework.

Here, we offer a unified and intuitive explanation for all three puzzles. In particular, we show that the dispersion, IV, and credit risk effects naturally arise in equilibrium and are inherently linked. The three effects emerge through the interaction of investor aversion to shocks in economic growth rates and firm cash flow timing. Specifically, low expected dividend growth firms derive their values primarily from short-run cash flows. Whereas
such firms exhibit low sensitivity to the aggregate risk factor, they have high exposure to idiosyncratic shocks, leading to high dispersion, IV, and credit risk levels. However, because these firms have low exposure to economic growth shocks, they have small loadings on the economic growth risk factor and thus yield low expected returns. In contrast, firms with values weighted towards long-run cash flows have greater exposure to aggregate risk and thus yield high expected returns. Such firms are relatively insensitive to idiosyncratic cash flow shocks and are characterized by low dispersion, IV, and credit risk levels. Thus, the negative correlation between expected return and dispersion, IV, and credit risk is a manifestation of cross-sectional differences in exposures to idiosyncratic and systematic sources of risks.

From an investment perspective, we simulate payoffs from trading strategies based on the dispersion, IV, and credit risk effects in our economy. For instance, the credit-risk-based trading strategy amounts to taking long (short) position in the decile of investable stocks having the lowest (highest) credit risk. The resulting monthly payoffs are 0.45%, 0.46%, and 0.38% for the dispersion, IV, and credit risk strategies, respectively. Investment payoffs are statistically and economically larger for small market capitalization stocks.

The three essential ingredients underlying our model are (i) the stochastic differential utility of Duffie and Epstein (1992) with elasticity of intertemporal substitution equal to unity, (ii) joint dynamics of consumption and dividend growth rates per the one-factor economy of Bansal and Yaron (2004), and (iii) a Wright–Fisher process for the dividend share dynamics. Whereas the first two ingredients are economy-wide, the third ingredient is the key to our micro modeling, as it allows us to split the aggregate dividend into firm-
level streams and formally model the cross section of firms. Thus, we successfully merge the long-run risk literature developed by Bansal and Yaron (2004) with the shares-based equilibrium cross-sectional literature developed by Menzly, Santos, and Veronesi (2004) and Santos and Veronesi (2006). Moreover, our proposed resolution of asset pricing puzzles is straightforward; stocks with higher dispersion, credit risk, and IV have smaller loadings on an economic growth hedge portfolio, thus delivering smaller payoffs.

In sum, recent work employs the long-run risk framework to offer explanations for aggregate and cross-sectional asset pricing puzzles including the equity premium, the risk-free rate, and the excess volatility puzzles (Bansal and Yaron (2004)), the value premium (Kiku (2006), Bansal, Kiku, and Yaron (2007), and Hansen, Heaton, and Li (2008)), as well as the momentum effect (Avramov and Hore (2008) and Zurek (2007)). Here, we take the long-run risk literature forward in both methodology and substantive findings. We derive formal cross-sectional asset pricing restrictions based on recursive dynamics and are able to express our model through an intuitive ICAPM relationship. Moreover, our model simultaneously resolves three apparently counter intuitive asset pricing regularities, the dispersion, IV, and credit risk effects.

The remainder of the paper is organized as follows. Section (1) introduces our economy at the aggregate and firm levels and develops asset pricing results. We examine the dispersion, IV, and credit risk effects in Section (2). Section (3) concludes.
1 The Model

Our economic setup is inspired by the long-run risk framework originated with Bansal and Yaron (2004). The long-run risk literature hinges on an interaction of non-time-separable preferences and persistent consumption growth. In such economies, a shock to expected consumption growth affects investors’ contemporaneous utility through the value function, which is embedded in current utility with non-time separable preferences of Duffie and Epstein (1992).\footnote{Proving the existence of a persistent consumption growth is a challenge faced by the long-run risk literature. Empirically, Bansal, Kiku, and Yaron (2007) find substantial evidence for a predictable component in consumption growth (also see Bansal and Yaron (2004), Kiku (2006), and Hansen, Heaton, and Li (2008)). Theoretically, Hansen and Sargent (2008) show that within the robust control framework, investors operate as if the economy has persistent consumption growth when faced with uncertainty about the true nature of the consumption growth process. Moreover, Kaltenbrunner and Lochstoer (2007) show that long-run risk can arise in an economy with i.i.d. production through optimal consumption smoothing by investors with Epstein–Zin (1989) preferences.} This mechanism introduces additional volatility into the pricing kernel, which is essential to satisfy the Hansen–Jagannathan (1991) bounds as it generates volatility and risk premia that match their empirical counterparts.

Our paradigm has two essential ingredients pertaining to dynamics of aggregate- and firm-level fundamentals. The aggregate economy is a particularly simple version of Bansal and Yaron (2004), where the agent is exposed to a single persistent shock that serves as a common time-series trend. This shock affects both expected consumption and dividend growth rates. As in Bansal and Yaron (2004), consumption growth occurs at a slower pace than dividend growth, thus capturing the insight of Abel’s (1999) levered economy. Furthermore, we focus on a special case of recursive preferences by setting the elasticity of intertemporal substitution (EIS) to unity. This special case allows us to derive explicit solutions for price-dividend ratios and return dynamics. Our inferences are insensitive to
the choice of the EIS parameter.\(^2\)

In the cross section, we enforce a strict adding-up constraint, where firm-level dividends add up to the aggregate dividend. The firm dividend share follows the tractable Wright–Fisher process which, under parameter restrictions similar to the Cox, Ingersoll, and Ross (1985) process, is bounded between zero and one. We derive closed-form price-dividend ratios for individual assets from which we formulate a tractable version of Merton’s (1973) ICAPM, where the agent is rewarded for exposure to a stochastic expected consumption growth shock via the log-recursive utility function. The combination of preferences and simplified share processes ultimately delivers key asset-pricing quantities including firm-level price-dividend ratio, risk premium, beta, idiosyncratic volatility, dispersion, and credit risk, all of which exhibit economically significant magnitudes which match the empirical evidence.

1.1 Consumption and Dividend Growth Rates

Our processes for aggregate consumption and dividend growth rates are the continuous-time versions of the discrete-time processes employed by Bansal and Yaron (2004).\(^3\) In

\(^2\)Empirically, there is a wide debate in the macroeconomic literature about the value of EIS. Kandel and Stambaugh (1991) focus on EIS close to zero. Hansen and Singleton (1982), Attanasio and Weber (1989), Vissing-Jorgensen (2002), and Guvenen (2006) argue that EIS is greater than 1, while Hall (1988) and Campbell and Mankiw (1989) argue that EIS is less than 1. Yogo (2004) estimates the EIS parameter in several countries and cannot reject EIS=1 in 11 of 13 countries.

\(^3\)We adopt a model with homoskedastic shocks to dividend growth, consumption growth, and the aggregate growth rate. Bansal and Yaron (2004) investigate both homoskedastic and heteroskedastic cases. The homoskedastic model is simpler and sufficient for our purposes. Furthermore, without altering the spirit of the model dynamics from the Bansal and Yaron (2004) economy, we multiply the stochastic component of expected consumption growth by \(\lambda\) instead of scaling the time-varying portion of expected dividend growth.
particular, consumption and dividend growth rates are formulated as

\[
\frac{dC_t}{C_t} = (\mu_C + \lambda X_t)dt + \sigma_C dW_C, \tag{1}
\]
\[
\frac{dD_t}{D_t} = (\mu_D + X_t)dt + \sigma_D dW_D, \tag{2}
\]
where \(X_t\) follows the mean-reverting dynamics

\[
dX_t = -\kappa X_t dt + \sigma_X dW_X. \tag{3}
\]

The common component \(X_t\) reverts back to zero, with the speed of mean reversion governed by the parameter \(\kappa\). Shocks to the time-varying component of aggregate economic growth, \(X_t\), propagate through several periods of consumption and dividend growth rates due to slow mean reversion, which gives rise to long-run risk. Since expected consumption growth is typically less volatile than expected dividend growth we allow for a differential effect of \(X_t\) on consumption and dividend growth rates, with the parameter \(\lambda\) controlling the relative strength of shocks. The processes \(dW_C, dW_D,\) and \(dW_X\) are assumed to be uncorrelated for simplicity of presentation.\(^4\)

\(^4\)Introducing correlation between the consumption and dividend processes leads to an additional term in expected returns taking the form \(\rho \sigma_C \sigma_D \gamma\), which is the familiar expected return in an \(i.i.d.\) growth economy with CRRA preferences. This term is economically small for reasonable values of risk aversion \(\gamma\) which gives rise to the equity premium puzzle found in economies with CRRA preferences and simple consumption dynamics (e.g. Mehra and Prescott (1985)). We assume zero correlation to simplify the model and focus attention on the effects of long-run risk. Our inferences are not qualitatively affected by this assumption.
1.2 Investor Preferences

The representative investor is endowed with stochastic differential utility (SDU) of Duffie and Epstein (1992) with EIS equal to one. The normalized aggregator in this case is given by

\[ f(C, J) = \beta(1 - \gamma)J \left[ \log C - \frac{\log((1 - \gamma)J)}{1 - \gamma} \right], \tag{4} \]

where \( J \) is the value function, \( \gamma \) is the risk aversion, and \( \beta \) is the time preference parameter.

1.3 Aggregate Asset Pricing

Given preferences and aggregate consumption and dividend growth rates, the investor can solve the price of aggregate claims and the corresponding equity premia. We present asset pricing quantities for the aggregate setup in the proposition below and refer the reader to Hore (2008) for complete derivation.

**Proposition 1.** For an agent exposed to consumption and dividend dynamics formulated in (1)-(3) and preferences given by (4), the following hold true:

- The value function is given by

\[ J(C_t, X_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \exp \left( \frac{\lambda (1-\gamma)}{\kappa + \beta} X_t + \frac{1-\gamma}{\beta} \left[ \mu_C - \frac{1}{2} \left( \frac{\lambda^2 (1-\gamma)\sigma_x^2}{\kappa + \beta} \right) \right] \right) . \tag{5} \]

- The pricing kernel is given by

\[ \frac{d\Lambda}{\Lambda} = -r_t^f dt - \gamma \sigma_C dW_C - \frac{(\gamma - 1)\lambda}{\kappa + \beta} \sigma_X dW_X, \tag{6} \]

\[ r_t^f = \beta + \mu_C + \lambda X_t - \gamma \sigma_C^2. \tag{7} \]
where \( r^f_t \) is the risk-free rate.

- The price-dividend ratio is

\[
\frac{P_t}{D_t} = G(X_t) = \int_t^\infty S(X_t, \tau)ds \quad [\tau = s - t],
\]

where

\[
S(X_t, \tau) = \exp(P_1(\tau)X_t + P_2(\tau)),
\]

with \( P_1 \) and \( P_2 \) given by,

\[
P_1(\tau) = \frac{1 - \lambda}{\kappa} (1 - e^{-\kappa \tau}), \quad (10)
\]

\[
P_2(\tau) = a\tau + b(e^{-\kappa \tau} - 1) + c(1 - e^{-2\kappa \tau}), \quad (11)
\]

\[
a = \left( \mu_D - \mu_C + \gamma \sigma_C^2 - \beta + \frac{1 - \lambda}{\kappa} \sigma_X^2 \left( \frac{1 - \lambda}{2\kappa} - \frac{(\gamma - 1)\lambda}{\kappa + \beta} \right) \right), \quad (12)
\]

\[
b = \frac{(1 - \lambda)\sigma_X^2}{\kappa^2} \left( \frac{1 - \lambda}{\kappa} - \frac{(\gamma - 1)\lambda}{\kappa + \beta} \right), \quad (13)
\]

\[
c = \frac{(1 - \lambda)^2\sigma_X^2}{4\kappa^3}. \quad (14)
\]

- The instantaneous risk premium and volatility of return are given by

\[
\mu_t = \frac{\lambda(\gamma - 1)\sigma_X^2}{\kappa + \beta} G_X, \quad (15)
\]

\[
\sigma_t = \sqrt{\sigma_D^2 + \left( \frac{G_X}{G} \sigma_X \right)^2}. \quad (16)
\]

- \( \mu_t \) is increasing in \( X_t \).

\(^5\) The transversality condition is satisfied if \( a < 0 \).
Our price-dividend ratio departs from the traditional exponentially affine form. The non-linearity of the log price-dividend ratio induces stochastic volatility in the market return which commands a time-varying equity premium. Since $P_1(\tau) > 0$, a positive shock to expected aggregate dividend growth causes both the price-dividend ratio and expected excess return to increase. Simulations show that the stochastic volatility arising from long-run risk – the second shock in the pricing kernel due to Duffie–Epstein preferences and the stochastic growth rate – is substantial with an average estimate of 15.68% in the post-war US economy. This priced volatility is large enough to generate an equity premium of 9.55% while maintaining a risk-free rate at 1.8%. Thus, our aggregate economy addresses the prominent equity premium, risk-free rate, and excess volatility puzzles. This time-series aspect of return characteristics is absent from the one-factor economy of Bansal and Yaron (2004), as they posit the price-dividend ratio to be exponentially affine in the growth rate. In their two-factor economy, the time-variation in risk premia emerges from the stochastic volatility of consumption growth. Since the log price-dividend ratio here is non-linear in the growth rate, expected return is time-varying even without the second channel.

We can gain insight on the long-run risk mechanism through a closer examination of aggregate market valuation. Recall from equation (8) that $\frac{P_t}{D_t} = G(X_t) = \int_t^\infty S(X_t, \tau)ds$. Figure (1) displays the shape of the $S(X_t, \tau)$ function. From this graph, we can study how much value is derived from expected future market dividends at various horizons. For example, at a 25-year horizon, $S(X_t, \tau)$ is equal to about 0.5. This implies that the expected market dividend in 25 years adds $0.50 to the current market value (normalizing the current market dividend to $1.00). Clearly, a substantial portion of the aggregate market

---

6Bansal and Yaron (2004), Constantinides and Ghosh (2008), and Hore (2008), among others, show that $\lambda < 1$, so $P_1(\tau)$ is strictly positive.
value is derived from the short- and intermediate-term cash flows. An interesting feature of the function is the hump shape which is attributable to differences in discount rates for dividends at different horizons. Over short horizons, expected dividend growth is higher than the discount rate leading to a dividend value larger than $1.00. At long horizons, expected dividend growth is overwhelmed by high discount rates. Hence, the present value of dividends at long horizons is relatively small.

Long-run cash flows are riskier than short-run cash flows in our economy, thus commanding higher risk premia. Figure (2) exhibits the investors’ required discount rate on cash flows at different horizons. For example, at a 25-year horizon the discount rate is about 11%, suggesting that investors are discounting the expected market dividend in 25 years (about $8.00 per $1.00 of current market dividend) at a rate of about 11% per year to arrive at its current value \( (\$8 \times \exp(-0.11 \times 25) = \$0.50) \).

Why do agents require large premia on long-run cash flows? Investors endowed with the SDU of Duffie and Epstein (1992) prefer earlier resolution of uncertainty about future utility if \( \gamma > 1 \) with unit IES.\(^7\) Even with persistent consumption and dividend growth rates, the magnitude of short-run cash flows is relatively certain since changes in growth rates have little effect on short-run cash flows. As a result, investors view short-run cash flows as relatively safe and discount these cash flows at a rate only slightly above the risk-free rate. However, much of the uncertainty surrounding long-run cash flows remains unresolved for a long period. Thus, investors discount long-run cash flows heavily giving rise to long-run risk premia.

\(^7\)In the general case, the relation becomes \( \gamma > \frac{1}{\psi} \).
1.4 Cross Section of Firms

We consider a cross section of firms that aggregates, by construction, to the economy derived above. We explicitly model the firm dividend share, the contribution of a particular firm to the aggregate dividend, in the spirit of Menzly, Santos, and Veronesi (2004) and Santos and Veronesi (2006) (Cochrane, Longstaff, and Santa-Clara (2008) endogenize their share process). Below, we introduce the dividend share process and then discuss firm-level cash flows.

1.4.1 The Dividend Share Process

We assume that there are \( n \) firms in our economy with the \( i \)-th firm contributing a portion \( \theta_t \) of the aggregate dividend at time \( t \), where \( \theta_t \) is defined to be the dividend share of firm \( i \) at time \( t \). Thus, if the aggregate dividend is \( D_t \) at time \( t \), the \( i \)-th firm contributes \( D_t^i = \theta_t D_t \), whereas the other \( n-1 \) firms contribute a total of \( (1-\theta_t)D_t \) such that the total aggregate dividend is \( D_t \). This holds trivially for two stocks, but we can always take the second stock as the composite of all firms in the economy, modulo the \( i \)-th stock. Therefore, a two-firm economy results for asset \( i \) are equivalent to those based on an \( n \)-firm economy.

The dividend share process \( \theta_t \) must satisfy the restriction that at all \( t \), \( \theta_t \in [0,1] \). We employ the Wright–Fisher (WF) process which captures this property.\(^8\) The WF process satisfies the SDE

\[
d\theta_t = \alpha(\bar{\theta} - \theta_t)dt + \delta\sqrt{(1-\theta_t)}\theta_t dW_t, \tag{17}
\]

where \( \alpha, \bar{\theta}, \) and \( \delta \) are firm-specific parameters underlying the dividend share process. Ob-

\(^8\)Wright–Fisher processes date back to genetics literature by Fisher (1930) and Wright (1931). See Crow and Kimura (1970) for examples and further discussion of Wright–Fisher processes.
serve from the diffusion term $\delta \sqrt{(1 - \theta_t)\theta_t}$ that the dividend share process is defined only between zero and one. In fact, if $\theta_0 \in (0, 1)$, then as long as $2\alpha\bar{\theta} > \delta^2$ (see Karlin and Taylor (1981), pp 239-241), the boundary points zero and one are unattainable within finite time. The diffusion term is somewhat similar to the square root process of Cox, Ingersoll, and Ross (1985), except the square root process does not have the extra $(1 - \theta_t)$ component that creates a boundary point at one. If the restriction $2\alpha\bar{\theta} > \delta^2$ is violated, we propose an absorbing barrier at each of the boundaries, mimicking firm bankruptcy and dissolution if $\theta_t = 0$.

The drift term $\alpha(\bar{\theta} - \theta_t)$ induces mean reversion in the firm dividend share. In particular, firm $i$ has a long-run mean level of dividend share $(\bar{\theta})$ and its dividend share fluctuates around the long-run mean with $\alpha$ controlling the speed of mean reversion. The mean-reversion component is essential to guarantee that no firm eventually takes over the economy. Of course, for every firm $i$, the remainder of the economy contributes a dividend share equal to $1 - \theta_t$, which also follows a mean-reverting WF process with long-run mean of $1 - \bar{\theta}$.

Following Menzly, Santos, and Veronesi (2004), we refer to $\bar{\theta}/\theta_t$ as the “relative share” of the firm. The relative share is the long-run expected dividend share of the firm as a proportion of its current dividend share. Several of our model implications depend on dividend share $\theta_t$ through the relative share $\bar{\theta}/\theta_t$.

To briefly illustrate the relative share characteristic, suppose that a firm currently pays 1% of the total market dividend but is expected to exhibit high dividend growth so that it eventually pays 5% of the overall market dividend. The dividend share of this firm
is 0.01 and the relative share is 5 \((\frac{5\%}{1\%})\). Consider another firm currently paying 1% of the market dividend but is expected to eventually pay only 0.2% of the market dividend. This firm has a dividend share of 0.01 and a relative share of 0.2 \((\frac{0.2\%}{1\%})\). By definition, the aggregate market always has dividend share and relative share equal to one, i.e. \(\theta_t = \bar{\theta} = 1\).

Armed with the dividend share process (17), we can derive firm-level fundamentals.

### 1.4.2 Firm-Level Dividend Growth

We assume that the dividend share process is uncorrelated with shocks to the consumption and dividend growth rates as well as the underlying expected growth rate (i.e., \(\text{Cov} \left( d\theta_t, \frac{dC_t}{C_t} \right) = 0, \text{Cov} \left( d\theta_t, \frac{dD_t}{D_t} \right) = 0, \text{and} \ Cov \left( d\theta_t, dX_t \right) = 0\)). Zero correlations imply that the dividend share does not covary with the pricing kernel, hence no asset in this economy hedges against transient consumption shocks from \(dW_C\). In the parlance of Menzly, Santos, and Veronesi (2004) and Santos and Veronesi (2006), the cash flow betas of all securities are zero in our economy. Nevertheless, as we show below, firm relative shares interact with the persistent component of consumption growth to play a powerful role in determining time-varying exposures to aggregate risk (loadings on an economic growth factor) through the non-linearities in the model.

A direct application of Ito’s Lemma to the firm dividend, \(D_t^i = \theta_t D_t\), gives

\[
\frac{dD_t^i}{D_t^i} = \left( \alpha \left( \frac{\bar{\theta}}{\theta_t} - 1 \right) + \mu_D + X_t \right) dt + \delta \sqrt{\frac{1 - \theta_t}{\theta_t}} dW_\theta + \sigma_D dW_D. \tag{18}
\]

Observe that the firm-level cash flows inherit properties from the aggregate economy. The
dividend growth rates of all firms in the economy share an aggregate dividend shock \( \sigma_D \), as well as expected aggregate dividend growth \( \mu_D + X_t \). However, the firm’s expected dividend growth also depends on the firm’s relative share \( \bar{\theta}_t \). To illustrate, assume that firm \( i \)’s dividend share is below its long-run mean, i.e. \( \bar{\theta} > \theta_t \), at time \( t \). Then firm \( i \) will likely experience high dividend growth in order to converge toward its long-run mean \( \bar{\theta} \). In this case, \( \bar{\theta}_t > 1 \), and the firm’s expected dividend growth is higher than the expected aggregate dividend growth. Similarly, if the firm pays higher dividend than its long-run mean, i.e. \( \bar{\theta} < \theta_t \), the firm’s dividend share is expected to decrease to its long-run mean \( \bar{\theta} \), and moreover the firm’s expected dividend growth will be lower than the aggregate expected dividend growth. When dividend share is equal to its long-run mean, i.e. \( \bar{\theta} = \theta_t \), the firm’s expected dividend growth is the same as the expected aggregate dividend growth. Still, the firm-level dividend growth is more volatile than aggregate-level dividend growth because of the extra noise attributable to \( dW_{\theta} \).

Even in the absence of explicit interaction between the diffusion components of dividend share of firm \( i \) and aggregate quantities, the dividend share creates cross-sectional variation in expected dividend growth through relative share – firms with high relative share (i.e. high \( \bar{\theta}_t \)) will have higher expected dividend growth than firms with low relative share. This cross-sectional distinction in expected dividend growth interacts with differences in long-run risk across cash flows at different maturities to create differences in asset prices and risk premia. As we show below, this interaction ultimately explains the apparently counter-intuitive relations between expected stock return and (i) dispersion, (ii) IV, and (iii) credit risk.

15
1.4.3 Firm Valuation

The price-dividend ratio for firm \( i \) is given by (derivation is in Appendix (A.1))

\[
\frac{P_i^t}{D_t^i} \equiv G^i(X_t, \theta_t; \bar{\theta}, \alpha) = \int_t^\infty S(X_t, \tau) E_t[\theta_s] ds \quad [\tau = s - t]
\]

\[
= \int_t^\infty S(X_t, \tau) \left[ e^{-\alpha \tau} + \frac{\bar{\theta}}{\theta_t} (1 - e^{-\alpha \tau}) \right] ds.
\]

(19)

The dividend share process is independent of the aggregate-level processes, hence \( E_t[\Lambda_s D_s] = E_t[\Lambda_s D_s] E_t[\theta_s] \), where the first conditional expectation delivers the aggregate price-dividend ratio.\(^9\) The firm price-dividend ratio is a non-linear function of the stochastic component of the aggregate growth rate, \( X_t \), and the firm relative share. All firms share a component in their price-dividend ratio attributable to the aggregate price-dividend ratio through \( S \) (see Proposition 1), whereas the firm-specific effect propagates through the relative share, \( \frac{\bar{\theta}}{\theta_t} \). Since the autoregressive coefficient of the mean-reverting dividend share process (\( \alpha \)) is greater than zero, equation (19) indicates that the firm price-dividend ratio is increasing in relative share. We also show in (18) that expected dividend growth of firm \( i \) increases if \( \frac{\bar{\theta}}{\theta_t} \) increases. Thus, the high expected dividend growth of firm \( i \) leads to an increase in firm value relative to its current dividend.

The firm-level price-dividend ratio can also be related to the market price-dividend ratio by

\[
\frac{P_i^t}{D_t^i} = \frac{P_t}{D_t} + \left( \frac{\bar{\theta}}{\theta_t} - 1 \right) \int_t^\infty S(X_t, \tau) (1 - e^{-\alpha \tau}) ds.
\]

(20)

Notice that if the firm’s dividend share \( \theta_t \) equals \( \bar{\theta} \), i.e. the dividend share is at its long-run

\(^9\)For \( \alpha > 0 \), integrability of (19) is trivially satisfied due to the transversality condition of \( S(X_t, \tau) \).
mean, then the firm’s dividend-price ratio will be exactly the same as that of the market. If the firm’s dividend share is below (above) its long-run mean such that \( \theta_t > 1 \) (\( \theta_t < 1 \)), then the firm’s expected dividend growth is higher (lower) than aggregate expected dividend growth, which pushes firm-level price-dividend ratio higher (lower) than the aggregate price-dividend ratio. However, high relative share also implies that the firm’s dividends are weighted towards the long-run, and agents exposed to long-run risk discount these cash flows more heavily, which moderates the increase in firm-level price-dividend ratio.

1.5 Cross-Sectional Instantaneous Return Characteristics

The firm-level instantaneous excess return follows a process that is determined in equilibrium by various shocks, some are priced (systematic shocks that co-vary with the pricing kernel) whereas some are unpriced (idiosyncratic risks). In particular, the instantaneous excess return \( R_i^t \) follows the dynamics (derivation is in Appendix (A.2))

\[
dR_i^t = \mu_i^t dt + \sigma_D dW_D + \frac{G^i_X}{G^i} \sigma_X dW_X + \delta \sqrt{1 - \frac{\theta_t}{\theta_t}} \left( 1 + \frac{\theta_t G^i_\theta}{G^i} \right) dW_\theta,
\]

where

\[
\mu_i^t = -Cov \left( \frac{dP_i^t}{P_i^t}, \frac{d\Lambda}{\Lambda} \right) = \frac{\lambda(\gamma - 1)}{\kappa + \beta} \sigma^2_X \frac{G^i_X}{G^i}.
\]

The first volatility term in (21) emerges due to aggregate dividend risk which is shared by all firms in the economy, but the agent is not compensated for this risk as a result of the simplifying assumption that consumption and dividend growth rates are uncorrelated. The second volatility term is priced and is attributable to the stochastic nature of aggregate growth rates, which induces long-run risk in the pricing kernel per Duffie–Epstein preferences. Indeed, the expected excess return in (22) is compensation for exposure to the
long-run risk shock in the pricing kernel. The third volatility component is firm-specific, and arises from shocks to dividend share leading to changes in the firm-level price-dividend ratio and expected dividend growth. This third component is uncorrelated with all other shocks and is firm-specific, thus we define it to be idiosyncratic volatility.

The relative share has thus far characterized firm-level dividend growth and price-dividend ratio. Next, we show the relation between relative share and expected return. We first establish the following result

**Theorem 1.** *Firm-level priced volatility* \( G \) *is increasing in* \( \theta \) *if* \( \alpha > 0 \) *and* \( \kappa > 0 \).

Theorem (1) indicates that under very mild regularity conditions (the autoregression parameter is positive for aggregate growth and dividend share), the priced volatility element increases in relative share. Since the agent is compensated for taking on exposure to this risk, expected return increases in relative share. Figure (3) shows the positive relation between expected stock return and dividend share and the proof of Theorem (1) is in Appendix (A.2).

The basic intuition underlying the positive relation between expected return and relative share is as follows. With unit EIS, an agent endowed with Duffie–Epstein SDU and \( \gamma > 1 \) prefers early resolution of uncertainty about the growth rate of consumption which affects future utility. This agent requires more compensation when the resolution of uncertainty is slower. Firms with high relative share will reach their long-run mean dividend share far into the future, and expected cash flows from these firms are heavily dependent on expected dividend growth rates. As a result, uncertainty about the cash flows of high relative share firms remains unresolved for an extended period of time and these firms are highly exposed
to aggregate risk (i.e., fluctuations in the pricing kernel). Therefore, an agent holding a claim to the cash flows of a high relative share firm requires a high expected return. Even as the expected returns of these firms are higher in equilibrium, high relative shares also give rise to high price-dividend ratios, just as in Menzly, Santos, and Veronesi (2004).

1.5.1 Conditional Beta Pricing Representation

Below we develop a version of Merton’s (1973) ICAPM with a single factor hedging against economic growth shocks. Typically, the ICAPM is written as a linear factor model with the aggregate wealth portfolio and additional intertemporal hedge portfolios. However, we zero out the correlation between dividend and consumption growth shocks to concentrate on long-run risk implications, thereby eliminating exposure to the total wealth portfolio. Our single-factor setup resembles Breeden’s (1979) equilibrium ICAPM except that exposure to economic growth replaces shocks to consumption growth. Investors who are averse to exposures to aggregate growth shocks would be willing to realize lower returns on assets that hedge against changes in expected growth. On the other hand, investors would require a large premium on assets with payoffs that are highly positively correlated with the aggregate growth rate. We show that price elasticity of individual assets with respect to the macroeconomic growth rate determines exposure to risk premia due to this stochastic growth rate. A high (low) price elasticity of an asset (relative to the hedge portfolio’s

\[ \text{E} \equiv \mu_{TW} = \sigma^2_C \gamma \text{ for all } t. \]  

Furthermore, firm expected returns become \( \mu_i = \gamma \sigma_C \sigma_D \rho + \frac{\lambda \sigma^2 - 1}{\kappa + \beta} \frac{\lambda^2 \sigma_X}{\sigma_D^2} \), so we can write expected firm return as \( \mu_i = \beta_{TW} \mu_{TW} + \beta_i \mu_t \), where \( \mu_t \) is the risk-premia due to exposure to the time-varying growth rate, \( \beta_{TW} = \frac{\sigma_D \delta C}{\sigma^2_C} \), and \( \beta_i \) is given in equation (24). Reintroducing correlation between consumption and dividend growth causes betas on the total wealth portfolio to be non-zero, but has no substantive impact on our cross-sectional results.
elasticity) leads to a higher (lower) loading on the risk factor, and hence higher (lower) expected return.

Observe from equations (15) and (22) that the expected excess return of firm $i$ is given by

$$
\mu^i_t = \lambda (\gamma - 1) \sigma^2_X \frac{G^i_X}{G^i} \\
= \frac{G^i_X}{G^i} G^i \mu_t \\
\equiv \beta^i \left( X_t, \frac{\bar{\theta}}{\bar{\theta}} \right) \mu_t.
$$

The expected return for each firm is linear in its conditional loading on the economic growth risk factor. This representation under Duffie–Epstein preferences is novel in the literature. Our model’s economic content lies in its predictions of the magnitude and evolution of firm betas and the factor risk premium. A firm’s conditional beta $\beta^i \left( X_t, \frac{\bar{\theta}}{\bar{\theta}} \right)$ is a non-linear function of the relative share $\frac{\bar{\theta}}{\bar{\theta}}$ and the aggregate economic growth state $X_t$, hence beta varies as relative share and economic growth evolve. The price of risk varies as the hedge portfolio’s exposure to long-run risk changes with shocks to aggregate economic growth.

Notice that the firm conditional beta takes the form

$$
\beta^i \left( X_t, \frac{\bar{\theta}}{\bar{\theta}} \right) = \frac{G^i_X}{G^i} G^i.
$$

Let us define the elasticity of hedge portfolio value with respect to the aggregate expected growth rate to be $\epsilon_X = G_X \frac{X}{G}$, so (roughly speaking) with a 1% shock to aggregate economic growth...
growth the hedge portfolio return is $\epsilon_X \%$. Similarly, the elasticity of the market value of firm $i$ is defined by $\epsilon_i = G_i \frac{X}{G}$. Thus, we can rewrite expected return as

$$\mu_i = \frac{\epsilon_i}{\epsilon_X} \mu_t.$$  \hspace{1cm} (25)

That is, the firm’s exposure to aggregate risk premia depends on the firm’s elasticity relative to the hedge portfolio’s elasticity. Firms with values that are highly sensitive to aggregate growth shocks exhibit high betas, and hence command high expected returns. Firms which are relatively immune to economic growth shocks exhibit low betas and thus deliver low expected returns.

Figure (4) presents a strong positive relation between relative share and conditional beta. In particular, high (low) relative share firms have primarily long-run (short-run) cash flows and have relatively high (low) correlation with returns on the hedge portfolio. To further illustrate, suppose that expected aggregate dividend growth is currently at its unconditional expectation level that we set at 6.5%. A negative 2% shock to the expected dividend growth from 6.5% to 4.5% results in a market return of -13.7%. A firm with relative share of 5 is more sensitive to economic growth and hence realizes a -15.7% return following the negative shock. On the other hand, a firm with relative share of $\frac{1}{5}$ is more immune to economic growth shocks so its return is higher at -10.8%. Investors require higher premia on high relative share firms that perform poorly when adverse economic conditions arise (in the case above, investors require a premium of 9.9% for the high relative share firm versus 6.8% for the low relative share firm).
Conditional betas evolve with changing expected economic growth. Figure (4) suggests that the cross-sectional dispersion of firm conditional betas is countercyclical. As expected economic growth falls, the conditional betas of high (low) relative share firms increase (decrease). Long-run cash flows become relatively more sensitive to economic growth shocks as expected growth declines. Concurrently, the importance of long-run cash flows to a firm’s value increases for high relative share firms and decreases for low relative share firms. This interaction leads to the observed countercyclicality in the cross-sectional dispersion of conditional betas.

1.6 Asset Pricing with Leverage

While the dispersion and IV effects can be analyzed in both levered and unlevered economies, the credit risk effect must be studied in the presence of leverage. Following Merton (1974), we assume the firm takes on debt with a face value of $B$ payable at time $T$. The firm defaults if its value is less than the face value of debt at maturity. Firm value follows the same process as the equity value $P_i^t$ from the no leverage case. At maturity, stockholders receive the residual firm value unpaid to bondholders. The value of firm equity follows by valuing a call option on firm assets,

$$V_i^t = E_t \int_t^T \frac{\Lambda_s}{\Lambda_t} D_s^t ds + E_t \left[ \frac{\Lambda_T}{\Lambda_t} \max \left( P_T^i - B, 0 \right) \right]. \quad (26)$$

The first equity-value component is the expected discounted dividend stream paid to stockholders prior to bond maturity. The second component is the expected discounted payoff to stockholders at bond maturity. If default occurs all firm assets are collected by bondholders.
Closed-form solutions to the equity value and expected return are unavailable in the presence of leverage. However, these quantities can be easily estimated through simulation (Appendix (B) describes the simulation details). Figure (5) graphs expected excess equity returns for levered firms versus their relative shares. As in the unlevered case, expected excess equity returns increase in relative share. Introducing leverage enhances the exposure of firm equity to long-run risk by magnifying the sensitivity of equity value to aggregate economic growth. Thus, expected excess equity return is higher for levered than unlevered firms (Figure (5) versus Figure (3)).

As shown earlier, the firm expected return obeys a conditional ICAPM in the unlevered case. In the presence of leverage, equity and debt expected returns still follow a conditional ICAPM. However, we are unable to derive analytical expressions for equity and debt conditional betas.\(^{11}\) Instead, we discuss the intuition behind the determination of levered equity betas and show estimated conditional equity betas using simulation.

Equity and debt betas vary with firm relative share, the aggregate economic growth state, firm leverage, and the time to maturity of the firm’s debt obligations. At any point in time, the firm’s overall conditional beta is equal to the weighted average of the conditional equity and debt betas. Here, we concentrate on the conditional equity beta since it is the core of our research design. Firm equity betas are typically higher than the overall

\[^{11}\text{Although a convenient analytical expression for levered equity beta is unavailable, we can show that levered equity returns obey the ICAPM. Firm levered equity returns follow the process } dR_t^Y = \mu_t^Y dt + \phi_X dW_X + O(\sqrt{dt}), \text{ where } \phi_X \text{ is the elasticity of firm equity value with respect to aggregate growth shocks. The } O(\sqrt{dt}) \text{ term contains all other diffusion terms (e.g. } dW_\theta \text{ and } dW_D \text{) that are uncorrelated with growth shocks. Then } \mu_t^Y = \frac{(\gamma - 1)\lambda}{\kappa + \beta} \sigma_X \phi_X = \phi_X \frac{\partial}{\partial X} \mu_t \equiv \beta_t^Y \mu_t. \text{ We do not have an analytical expression for } \phi_X, \text{ which prevents us from arriving at a closed-form solution for } \beta_t^Y. \text{ Levered equity beta is a time-varying function of aggregate economic growth, firm value, debt face value, time to debt maturity, and dividend share. A similar argument can be made for debt.}\]
firm betas. The value of debt is typically less sensitive to economic growth shocks than the value of equity, so leverage amplifies the effects of economic shocks on equity value as equity absorbs the majority of changes in the firm value. Thus, in the presence of leverage, the equity beta is higher than the beta of an otherwise identical unlevered firm, and expected equity return for the levered firm is higher than the unlevered firm’s expected return. Figure (6) confirms that levered equity betas are higher than unlevered betas in our simulation.

In the presence of leverage, conditional equity betas still increase in relative share. Figure (6) shows estimated conditional equity betas for firms with one year to debt maturity and a market debt ratio of 0.5. Introducing leverage increases the cross-sectional spread in conditional equity betas and, consequently, increases the cross-sectional spread in expected returns. In sum, higher relative share firms have higher conditional equity betas and expected equity returns for both levered and unlevered firms.

2 Dispersion, Idiosyncratic Volatility, and Credit Risk Effects

This section develops our model implications for the cross-sectional associations between expected return and (i) dispersion, (ii) IV, and (iii) credit risk. We ultimately confirm that our model resolves these three puzzling negative relations. The three effects are inherently linked and they naturally emerge in a conditional single factor model.
2.1 Dividend Dispersion

DMS document a negative relation between average stock return and the dispersion of analysts’ earnings forecasts. Our model predicts this negative relation. Briefly, low relative share firms have values arising primarily from short-run cash flows. Such firms exhibit relatively high cash flow growth volatility but little systematic risk exposure and hence low expected returns. Conversely, high relative share firms with low current cash flows but high expected growth exhibit less future cash flow growth volatility but substantial exposure to economic growth shocks, thus causing investors to require high returns.

Given the firm dividend growth process formulated in equation (18), dispersion is given by

\[
\text{Dispersion} = \left| \frac{\sqrt{\delta^2 \left( \frac{r_t}{\theta} - 1 \right) + \sigma_D^2}}{\alpha(r_t - 1) + \mu_D + X_t} \right|, \tag{27}
\]

where \( r_t = \frac{\bar{\theta}}{\theta_t} \). Here, dispersion measures the uncertainty about future dividend growth defined as the standard deviation of future dividend growth normalized by expected dividend growth. DMS examine the dispersion of analysts’ earnings forecasts, which can be viewed as a proxy for uncertainty in the underlying cash flows. In fact, DMS show a positive relation between dispersion in earnings forecasts and cash flow variability but dismiss a risk-based explanation based on cash flow uncertainty due to the negative relationship between expected return and dispersion. Our model provides a risk-based explanation for the expected return-dispersion relation.

Figure (7) plots dispersion as a function of relative share for simulated firms (simulation details are provided in Appendix (B)). The figure suggests that there is an inverse
relationship between relative share and dispersion. High relative share firms tend to have less uncertainty in future dividend growth than low relative share firms.

Our model predicts a negative cross-sectional relationship between expected return and dispersion for both levered and unlevered firms as depicted in Figures (8) and (9). The dispersion effect persists even if we control for size. In particular, we sort the cross section of firms into size terciles based on market value and plot firms within each size group. In Figures (8) and (9), red $\times$ signs (blue + signs) represent the largest (smallest) capitalization firms.

Table (1) shows the correlation between dispersion and expected return per market capitalization groups. There is a strong negative correlation unconditionally as well as conditional on size. Within each size group the negative relationship between expected return and dispersion persists, consistent with the empirical evidence uncovered by DMS. Our model also produces economically significant payoffs by implementing a dispersion strategy of buying low dispersion and selling high dispersion firms. As shown in Panel A of Table (2), the dispersion strategy produces an expected monthly return of 0.45% for levered firms and 0.19% for unlevered firms. Dispersion strategy profits persist within size terciles.

2.2 Idiosyncratic Volatility

AHXZ document a negative relation between IV and expected return in both US and international markets. While the IV effect is at odds with existing economic theory, it naturally arises within our setup. In fact, the IV effect is closely related to the dispersion effect.
As shown above, low relative share firms have high cash flow uncertainty and they exhibit high price sensitivity to idiosyncratic cash flow shocks. Thus, low relative share firms have high IV but low expected return due to their low exposure to systematic risk. High relative share firms have low dispersion and their values are more immune to idiosyncratic cash flow shocks leading to low IV.

The idiosyncratic component of the firm realized rate of return in (21) arises from firm-specific shocks to the firm dividend share. These dividend share shocks are uncorrelated with all aggregate-level shocks. Observe from equation (21) that the instantaneous variance of unlevered firm excess return can be expressed as

\[
\text{var}(R^i) = \sigma_D^2 + \left(\frac{G^i_X}{G^i}\right)^2 \sigma_X^2 + \delta^2 \frac{1 - \theta_t}{\theta_t} \left(1 + \frac{\theta_t G^i_u}{G^i}\right)^2.
\] (28)

The idiosyncratic component of firm return volatility is

\[
\delta \sqrt{\frac{1 - \theta_t}{\theta_t} \left(1 + \frac{\theta_t G^i_u}{G^i}\right)}.
\] (29)

Idiosyncratic volatility decreases in relative share under the following sufficient condition.

**Theorem 2.** Idiosyncratic volatility decreases in relative share under the restriction on the \((\bar{\theta}, \alpha)\)-plane \((r_t - 2\bar{\theta})(\sqrt{2\alpha M} - 1) > 1\) where \(r_t = \frac{\bar{\theta}}{\sigma_t^2}, M = \frac{|S(X_t, \tau)|_1}{|S(X_t, \tau)|_2}\) is independent of \(\alpha\), and \(\cdot\)\(_i\) represents the \(i\)-th norm.

The proof is in Appendix (A.3). A condition for Theorem (2) to hold is that \(\theta_t < \frac{1}{2}\), which occurs with near certainty given reasonable firm-level parameters and must obtain
for at least \( n - 1 \) of the \( n \) firms in the cross section. For a given \( \theta_t < \frac{1}{2} \), any \( (\bar{\theta}, \alpha) \) that satisfy the restriction imposed by Theorem (2) will give rise to a negative relation between relative share and IV. In fact, since \( S > 0 \), the 1-norm of \( S \) is much greater than the 2-norm and \( M >> 1 \), such that the above condition flexibly holds under a wide set of parameter values that we consider here.

Figure (10) plots idiosyncratic volatility as a function of relative share (simulation details are provided in Appendix (B)). Low relative share firms primarily derive their values from short-run cash flows and are therefore sensitive to fluctuations in firm-level dividend shares. Thus low relative share firms have higher IV. Intuitively, most of the value of low relative share firms arises from short-run dividends. An unexpected change in dividend share leads to relatively large revisions in expectations of short-run dividends. As a result, firm value is highly sensitive to dividend share shocks, which are idiosyncratic events. In contrast, high relative share firms derive most of their value from long-run dividends. Expectations of long-run dividends are less sensitive to current dividend share shocks. Idiosyncratic shocks to dividend share therefore lead to smaller changes in firm value for high relative share firms. Instead, high relative share firms have much higher systematic risk levels due to their exposure to the aggregate growth rate.

Figures (11) and (12) demonstrate that, consistent with the empirical evidence, our model produces a negative relation between expected return and idiosyncratic volatility for both unlevered and levered firms. High relative share firms have large exposures to long-run risk, leading to high expected return but low idiosyncratic volatility arising from current dividend shocks. On the other hand, the value of low relative share firms is more
heavily affected by idiosyncratic changes in short-run cash flows. Hence these firms have high idiosyncratic volatility but low expected return due to lower exposure to systematic risk. Further, Figures (11) and (12) show that the IV effect persists even after controlling for size. We sort firms into size terciles based on market value and then plot firms within each size group. Red × signs (blue + signs) represent the largest (smallest) capitalization firms. The IV effect indeed arises within each size group, consistent with the empirical evidence in AHXZ.

Table (1) reports large negative correlation coefficients between estimated IV and expected return in our model. From an investment perspective, a trading strategy that buys low IV stocks and sells high IV stocks produces a monthly expected return of 0.46% for levered firms and 0.17% for unlevered firms, as shown in Panel B of Table (2). The profitability of the IV based investment strategy remains within size terciles.

In sum, the apparently puzzling dispersion and IV effects are attributable to cross-sectional differences in exposure to systematic risk. Our explanation of the dispersion and IV effects is substantially different from the explanation of Johnson (2004) who attributes these effects to an option-pricing result in a Merton (1974) framework. Increasing IV leads to higher current levered equity values, leading to lower expected returns as future expected payoffs remain unchanged. While Johnson’s explanation arises from exposure to idiosyncratic risk, our explanation is rooted in exposure to varying level of priced and unpriced risks. Additionally, Johnson’s model relies on leverage to produce the dispersion and IV effects through option pricing results. Our model produces the dispersion and idiosyncratic volatility effects for both levered and unlevered firms, consistent with the
empirical evidence.

2.3 Credit Risk

Empirical work has uncovered a negative relation between credit risk and expected return (e.g. Dichev 1998, Avramov, Chordia, Jostova, and Philipov 2008, Campbell, Hilscher, and Szilagyi 2008). This evidence is quite surprising, as high credit risk firms are perceived to be riskier. The literature has typically offered non-risk-based explanations for the credit risk effect. However, we show that the credit risk effect arises naturally in our model due to cross-sectional differences in exposures to systematic and idiosyncratic risks. Specifically, we show that most defaults are primarily attributable to idiosyncratic risk which is unrewarded in equilibrium. On the other hand, firms with high exposure to long-run risk tend to default less often but earn high returns due to systematic risk exposure.

In the Merton (1974) framework, default occurs if the firm value drops below the face value of bonds at maturity, that is, if $V_T < B$. We estimate the probability of default by simulating the firm value through the bond maturity and then calculate the percentage of simulations in which the firm defaults. The simulation procedure is explained in more detail in Appendix (B).

Figure (13) graphs the one-year probability of default as a function of relative share for simulated firms. The evidence shows that default risk decreases in relative share. Indeed, most firms in the cross section have very small probabilities of default, even as our model produces a set of highly distressed firms. Low relative share firms exhibit high credit risk but typically default as a result of negative idiosyncratic events. Therefore, even when
low relative share firms have high credit risk, they are not perceived to be particularly risky since default events can be diversified away. Investors expect to be compensated for expected losses from default, but do not require a large risk premium to guard against idiosyncratic default events.

In contrast, high relative share firms have low credit risk. However, default events of high relative share firms are much more likely to be caused by negative aggregate economic growth shocks. Occasionally, the economy realizes large negative shock that induces investors to substantially diminish expectations about future consumption and dividend growth. These events are marked by “default waves.” In contrast to normal economic periods, many of the defaulting firms in such default waves have high relative share with \textit{ex ante} high credit quality. Despite the low probability of these events, investors require a large premium for taking on exposure to these events.

Figure (14) graphs expected return as a function of default probability. Investors require high expected return on low credit risk firms. Low credit risk firms have high levels of exposure to long-run risk (thereby commanding high risk premia) and are particularly likely to default, or at least have poor returns during bad economic times. On the other hand, firms with high probabilities of default typically have low exposure to long-run risk but have high IV. Investors require high enough expected return to compensate for expected losses in default events, but do not require a large premium for taking on exposure to defaults by low relative share firms.

From an investment perspective, Table (2) shows that a trading strategy that buys low
credit risk and sells high credit risk firms earns an expected monthly return of 0.38%. Distressed small firms tend to have higher probabilities of default than distressed large firms, and the expected return of the credit risk strategy in small firms is higher at 0.56% per month. Furthermore, consistent with the empirical evidence, we find that firms with the most extreme default probabilities have quite low expected returns. For example, the firms with the highest one percent of default probabilities have expected returns that are 0.70% per month lower than those of low credit risk firms.

3 Conclusion

We develop an intertemporal asset pricing model with time-varying risk loadings and risk premium in an economy where the representative agent is endowed with Duffie–Epstein preferences and exposed to persistent dividend and consumption growth rates. The economy is populated by a natural cross section of firms whose cash flows add up to the aggregate dividend. Our model suggests that the puzzling negative cross-sectional relations between expected stock return and analysts’ forecast dispersion, idiosyncratic volatility, and credit risk emerge in equilibrium due to the interaction between investor aversion to shocks in economic growth rates and firm cash flow timing. Specifically, low relative share firms have low expected dividend growth rates and market values that are weighted towards short-run cash flows, while high relative share firms have high expected dividend growth rates and values that are heavily influenced by long-run cash flows.

In an economy with aggregate risk governed by exposures to long-run risk, firms with
cash flows weighted towards the long run are riskier. As a result, high relative share firms, i.e. firms which will pay higher level of dividends far into the future, are highly exposed to aggregate economic growth shocks. Hence, investors require a large premium to hold such risky assets. These firms have low dispersion, IV, and credit risk since they have less cash flow uncertainty and their prices are relatively immune to idiosyncratic cash flow shocks. Conversely, low relative share firms naturally have higher cash flow uncertainty and price sensitivity to idiosyncratic events, leading to high dispersion, IV, and credit risk. At the same time, these firms are relatively insensitive to economic growth shocks leading to low systematic risk levels and thus low expected returns. Thus, the negative relations between expected returns and dispersion, IV, and credit risk arise as a result of cross-sectional differences in exposures to idiosyncratic and systematic sources of risks.

This paper contributes to the growing long-run risk literature, which offers explanations for a variety of aggregate and firm-level asset pricing puzzles, including the equity premium, risk-free rate, and excess volatility puzzles, as well as the momentum and value effects. Our contributions are both in methodology and substantive findings. Indeed, we derive formal cross-sectional asset pricing restrictions in a long-run risk setting summarized by an intuitive conditional single beta representation. Moreover, we demonstrate that the analysts’ forecast dispersion, idiosyncratic volatility, and credit risk effects are inherently linked through cross-sectional exposures to idiosyncratic and systematic sources of risks. Overall, our model generates aggregate- and firm-level return characteristics that match the empirical evidence and offers risk-based explanations to puzzling cross-sectional effects.
A Firm Valuation and Returns

The valuation of an unlevered firm is based on the traditional way of discounting future cash flows of firm \( i \) \( (D^i_t = \theta_t D_t) \) by the pricing kernel formulated in Proposition (1). Standard equilibrium arguments are applied to derive volatility and expected return.

A.1 Price-Dividend Ratio

The value of a firm is the expected discounted value of firm dividends,

\[
P^i_t = E_t \int_0^\infty \frac{\Lambda_s}{\Lambda_t} D^i_s ds
= E_t \int_0^\infty \frac{\Lambda_s}{\Lambda_t} D_s \theta_s ds.
\]  

(30)

In order to value the future dividends paid by firms, we need to compute \( E_t[\theta_s] = f(\theta_t, \tau) \), where \( \tau = s - t \). Applying Ito’s Lemma and Feynman–Kac to find \( E_t[df] \),

\[
\alpha(\bar{\theta} - \theta)f_\theta + \frac{1}{2}\sigma^2 \theta(1 - \theta)f_{\theta\theta} - f_\tau = 0,
\]  

(31)

which has a solution

\[
f(\theta_t, \tau) = E_t[\theta_s] = \theta_t e^{-\alpha \tau} + \bar{\theta}(1 - e^{-\alpha \tau}).
\]  

(32)

Notice that this is simply a weighted average of the current dividend share \( \theta_t \) and the long-run mean dividend share \( \bar{\theta} \), with \( f(\theta_t, \tau) = \theta_t \) at \( \tau = 0 \) and \( \lim_{\tau \to \infty} f(\theta_t, \tau) = \bar{\theta} \) as our expectation of \( \theta \) converges to its long-run mean as \( \tau \) grows large.
Then the price of security $i$ is

$$P_t^i = \frac{1}{\Lambda_t} E_t \int_t^\infty \Lambda_s \theta_s D_s ds$$

$$= \frac{1}{\Lambda_t} \int_t^\infty E_t[\Lambda_s \theta_s D_s] ds$$

$$= \frac{1}{\Lambda_t} \int_t^\infty E_t[\Lambda_s D_s] E_t[\theta_s] ds \quad \text{[independence]}$$

$$= \frac{1}{\Lambda_t} \int_t^\infty \Lambda_s D_s S(X_t, \tau) [\theta_s e^{-\alpha \tau} + \bar{\theta}(1 - e^{-\alpha \tau})] ds \quad [\tau = s - t]. \quad (33)$$

The price-dividend ratio of firm $i$ follows by dividing both sides by $D_t^i = \theta_t D_t$

$$\frac{P_t^i}{D_t^i} \equiv G^i(X_t, \theta_t, \tau; \bar{\theta}) = \int_t^\infty S(X_t, \tau) \left[ e^{-\alpha \tau} + \frac{\bar{\theta}}{\theta_t} (1 - e^{-\alpha \tau}) \right] ds. \quad (34)$$

### A.2 Returns

Individual firm excess return obeys

$$dR_t^i = \frac{dP_t^i + D_t^i dt}{P_t^i} - r_t^f dt$$

$$= \mu_t^i dt + \sigma_D dW_D + \frac{G_t^i}{G_t^i} \sigma_X dW_X + \delta \sqrt{\frac{1 - \theta_t^i}{\theta_t^i}} \left(1 + \frac{\theta_t^i G_t^i}{G_t^i} \right) dW_\theta,$$ \quad (35)

where $\mu_t^i$ is the expected excess return given by

$$\mu_t^i = -\text{Cov} \left( \frac{dP_t^i}{P_t^i}, \frac{d\Lambda}{\Lambda} \right)$$

$$= \frac{\lambda(\gamma - 1)}{\kappa + \beta} \sigma_X^2 \frac{G_t^i}{G_t^i}.$$

$$\quad (36)$$

We now establish conditions under which $\mu_t^i$ is increasing in relative share. First, define
relative share \( r_t = \frac{\theta_t}{\theta_i} \). The priced volatility component of total volatility of asset \( i \) is

\[
PV_t = \frac{G^i_X}{G^i} \sigma_X, \tag{37}
\]

where

\[
G^i_X = r_t \int_t^\infty S(X_t, \tau)P_1(\tau)(1 - e^{-\alpha \tau})d\tau + \int_t^\infty S(X_t, \tau)P_1(\tau)e^{-\alpha \tau}d\tau \tag{38}
\]

\[
G^i = r_t \int_t^\infty S(X_t, \tau)(1 - e^{-\alpha \tau})d\tau + \int_t^\infty S(X_t, \tau)e^{-\alpha \tau}d\tau. \tag{39}
\]

It is called priced volatility because \( PV_t \) is the only component of the price volatility that covaries with the pricing kernel. The following theorem shows that expected return is increasing in relative shares because \( PV_t \) is increasing in relative shares.

**Proof of Theorem 1:**

*Priced volatility is increasing in relative share if \( \alpha > 0 \) and \( \kappa > 0 \).*

**Proof:** Since \( G^i \) and \( G^i_X \) are both linear in \( r_t \), the sign of \( \frac{\partial PV_t}{\partial r_t} \) depends on the sign of

\[
c(r_t) = \int_t^\infty S(X_t, \tau)P_1(\tau)ds \int_t^\infty S(X_t, \tau)e^{-\alpha \tau}d\tau - \int_t^\infty S(X_t, \tau)P_1(\tau)e^{-\alpha \tau}d\tau \int_t^\infty S(X_t, \tau)ds. \tag{40}
\]

Substituting in \( P_1(\tau) = \frac{1-\lambda}{\kappa} (1 - e^{-\kappa \tau}) \) the sign of \( c \) is determined by\(^{12}\)

\[
\text{sgn}(c(r_t)) = \text{sgn} \left( \int_t^\infty S(X_t, \tau)ds \int_t^\infty S(X_t, \tau)e^{-(-\alpha + \kappa) \tau}d\tau - \int_t^\infty S(X_t, \tau)e^{-\alpha \tau}d\tau \int_t^\infty S(X_t, \tau)e^{-\kappa \tau}d\tau \right). \tag{41}
\]

In order to show \( \text{sgn}(c(r_t)) > 0 \), we need to show that

\[
\int_t^\infty S(X_t, \tau)ds \int_t^\infty S(X_t, \tau)e^{-(-\alpha + \kappa) \tau}d\tau > \int_t^\infty S(X_t, \tau)e^{-\alpha \tau}d\tau \int_t^\infty S(X_t, \tau)e^{-\kappa \tau}d\tau. \tag{42}
\]

\(^{12}\)Hore (2008) and Bansal and Yaron (2004) already show that \( \lambda < 1 \) in the post-war US economy.
We divide both sides of (42) by \((\int_t^\infty S(X_t, \tau)ds)^2\). Now, (42) reduces to
\[
\left(\int_t^\infty S'(X_t, \tau)e^{-(\alpha+\kappa)\tau}ds\right) > \left(\int_t^\infty S'(X_t, \tau)e^{-\alpha\tau}ds\right)\left(\int_t^\infty S(X_t, \tau)e^{-\kappa\tau}ds\right),
\]
where \(S'(X_t, \tau) = \frac{S(X_t, \tau)}{\int_t^\infty S(X_t, \tau)ds}\) is a particular density function since \(S > 0\).

Hence, (43) simply becomes equivalent to showing
\[
Cov(A, B) = E(AB) - E(A)E(B) > 0,
\]
where \(A = e^{-\kappa\tau}, B = e^{-\alpha\tau}\) under the density function \(S'\). Assume that for a \(\tau^*\), \(E(A) = e^{-\kappa\tau^*}\) and \(E(B) = e^{-\alpha\tau^*} + C\) where \(|C| < 1\). Since the random variables \(A\) and \(B\) only take values between zero and one, both \(E(A)\) and \(E(B)\) are between zero and one, and \(\exists\) unique \(\tau^* > 0\) and \(|C| < 1\), given \(\kappa > 0\) and \(\alpha > 0\), such that \(E(A) = e^{-\kappa\tau^*}\) and \(E(B) = e^{-\alpha\tau^*} + C\).

To show the above \(Cov(A, B) > 0\), we need to show that
\[
E\left[\left(e^{-\kappa\tau} - e^{-\kappa\tau^*}\right)\left(e^{-\alpha\tau} - (e^{-\alpha\tau^*} + C)\right)\right] > 0.
\]

Since the covariation of the first term in parentheses with the constant \(C\) is always zero, the above inequality is equivalent to showing
\[
E\left[(e^{-\kappa\tau} - e^{-\kappa\tau^*})(e^{-\alpha\tau} - e^{-\alpha\tau^*})\right] > 0.
\]

The random variable inside the expectation in the last inequality is always positive if \(\alpha > 0\) and \(\kappa > 0\). For \(\tau < \tau^*\), both \(e^{-\kappa\tau} - e^{-\kappa\tau^*}\) and \(e^{-\alpha\tau} - e^{-\alpha\tau^*}\) are positive, and for \(\tau > \tau^*\), both \(e^{-\kappa\tau} - e^{-\kappa\tau^*}\) and \(e^{-\alpha\tau} - e^{-\alpha\tau^*}\) are negative. □
A.3 Idiosyncratic Volatility

Since the Brownian motion component of the share process $dW_\theta$ does not covary with aggregate quantities, the volatility of return induced by the share process forms the idiosyncratic component of firm volatility.

Idiosyncratic volatility of asset $i$ is given by

$$IV^i_t = \delta \sqrt{\left(1 - \frac{1}{\theta_t}\right)\left(1 + \frac{\theta_t G_\theta}{G_i}\right)\int_t^\infty S(X_t, \tau) e^{-\alpha \tau} ds}$$

Notice the change of variable $\bar{\theta} = r_t$ which implies that $\frac{1}{\theta_t} = r_t$. By construction, $r_t > 1$.

Proof of Theorem 2:

Idiosyncratic volatility is decreasing in relative shares under the restriction on the $(\bar{\theta}, \alpha)$-plane $(r_t - 2\bar{\theta})(\sqrt{2\alpha M - 1}) > 1$ where $M = \frac{|S(X_t, \tau)|_1}{|S(X_t, \tau)|_2}$ is independent of $\alpha$ and $|\cdot|_i$ is the $i$-th norm.

Proof: The sign of the derivative $\frac{\partial IV^i_t}{\partial r_t}$ depends on whether the quantity

$$c(r_t) = \frac{\sqrt{\frac{\bar{m}}{\bar{g}} - 1}}{G + (r_t - 1) \int_t^\infty S(X_t, \tau)(1 - e^{-\alpha \tau}) ds}$$

is increasing or decreasing in $r_t$. By taking the derivative of $c(r_t)$ with respect to $r_t$, $\text{sgn}\left(\frac{\partial IV^i_t}{\partial r_t}\right)$ is determined by the sign of the quantity

$$\bar{c} = (2\bar{\theta} - r_t) \int_t^\infty S(X_t, \tau)(1 - e^{-\alpha \tau}) ds + \int_t^\infty S(X_t, \tau)e^{-\alpha \tau} ds.$$
Rewrite $\bar{c}$ as

$$\bar{c} = (2\bar{\theta} - r_t) \int_t^\infty S(X_t, \tau) d\tau + (r_t - 2\bar{\theta} + 1) \int_t^\infty S(X_t, \tau) e^{-\alpha \tau} d\tau$$

(50)

$$\leq (2\bar{\theta} - r_t) |S(X_t, \tau)|_1 + \frac{(r_t - 2\bar{\theta} + 1)}{\sqrt{2\alpha}} |S(X_t, \tau)|_2,$$

(51)

where the last inequality is due to the Cauchy–Schwartz inequality and $| \cdot |_i$ is the i-th norm. The success of (51) is to separate $\alpha$ from inside the integral, thus disentangling any non-linear dependence from other variables inside the integral. Now, we can enforce a condition for the idiosyncratic volatility result in the $(\bar{\theta}, \alpha)$-plane that is independent from the other variables in the system. Since, $S(X_t, \tau)$ is positive $\forall \tau$, the first norm is much greater than the second norm because the cross-terms are all positive. Thus, for $\bar{c} < 0$ in (51), all we need is to pick $\alpha$ such that

$$(r_t - 2\bar{\theta})(\sqrt{2\alpha} M - 1) > 1,$$

(52)

where $M = \frac{|S(X_t, \tau)|_1}{|S(X_t, \tau)|_2} >> 1$ for reasonable parameter values that we consider here. The first term $r_t - 2\bar{\theta} > 0$ for $\theta_t < \frac{1}{2}$. In fact, $r_t - 2\bar{\theta}$ is increasing as $\theta_t$ decreases. Fix $\theta_t < \frac{1}{2}$ and pick a $\bar{\theta}$. Then, one can pick $\alpha$ according to the above condition to ensure that $\bar{c} < 0$.\footnote{Notice, this holds true for the upper bound. It is possible that $\bar{c} < 0$ under looser restrictions.}

Simulations show that this bound can be easily achieved. \Box

### B  Model Simulation

We employ simulation experiments to investigate the returns of levered and unlevered equity and explore the interrelations between returns, dispersion, idiosyncratic volatility,
and credit risk. In this appendix, we give details of our simulation procedure.

For our base case simulation, we assume firm debt has one year to maturity. We model
debt in the framework of Merton (1974), so the firm is assumed to default if firm value is less
than the face value of debt at the time of debt maturity. For each firm in our simulation,
we set the face value of debt to 50% of the value of the firm at the time of issuance.\footnote{Since the face value rather than the market value of debt is set to 50% of the market value of the firm, firms have slightly different degrees of market leverage in our simulation. We set the face value of debt to a constant proportion of firm value for simplicity, since we must provide firm leverage as an input to the simulation and cannot estimate the market value of debt until we have the simulation results. These cross-sectional differences in leverage do not contribute to our results. In the simulation, firms with a high \textit{ex ante} probability of default will be assigned slightly lower leverage than low credit risk firms since high credit risk debt is more heavily discounted. Therefore, setting the face value of debt, rather than the market value of debt, equal to 50% of firm value biases the simulation against finding negative relationships between expected returns and dispersion, IV, and credit risk.}

The simulation procedure follows:

- We first draw 10,000 realizations of the aggregate economy for a one-year period. For each realization, we draw a parameter vector from the posterior distribution of aggregate economy parameters from Hore (2008). Given this set of parameters, we simulate a time-series of the processes $X_t$, $D_t$, and $\Lambda_t$ from time 0 to time $T$ using discretized versions of equations (2), (3), and (6). For our base case, we set $X_0 = 0$. Our results are robust to setting $X_0 = -\frac{\sigma X}{\sqrt{2\kappa}}$ or $X_0 = \frac{\sigma X}{\sqrt{2\kappa}}$. Results for these cases are available from the authors upon request.

- We create a cross section of 500 firms. For each firm, we draw random firm-level parameters that control the behavior of the firm’s dividend share. We draw $\alpha \in [0.05, 0.15]$ and $\delta \in [0.04, 0.10]$ from uniform distributions. Drawing $\alpha$ between 0.05 and 0.15 closely matches the empirical estimates of Menzly, Santos, and Veronesi (2004), and $\delta$ between 0.04 and 0.10 calibrates to the magnitude of idiosyncratic volatility observed in empirical data. We also randomly draw the beginning-of-period
dividend share $\theta_0$ and the long-run mean of dividend share $\bar{\theta}$. We take $\theta_0$ and $\bar{\theta}$ as the diagonal elements of a random matrix drawn from the Wishart distribution (the multivariate generalization of the $\chi^2$ distribution) with scale matrix $\frac{1}{4} \begin{pmatrix} .02 & .005 \\ .005 & .02 \end{pmatrix}$ and 4 degrees of freedom. The resulting draws of $\theta_0$ and $\bar{\theta}$ are strictly positive and each have mean 0.02. The draws of $\theta_0$ and $\bar{\theta}$ are positively correlated, so the cross section of relative share $\frac{\bar{\theta}}{\theta_0}$ is concentrated around one. The draws of $\theta_0$ and $\bar{\theta}$ produce reasonable cross-sectional distributions of dividend share and relative share.

- For each of the realizations of the aggregate economy, we draw a one-year time series of firm dividend share $\theta_t$ for each firm in the cross section using a discretized version of equation (17).

The aggregate processes $X_t$, $D_t$, and $\Lambda_t$ combined with the cross section of firm dividend shares $\theta_t$ completely describe the economy. Armed with aggregate and firm-level parameters and realizations of these processes, we examine firm characteristics including dispersion, idiosyncratic volatility, and credit risk as well as levered and unlevered returns. Following is our procedure to calculate these objects of interest.

- **Dispersion**: Dispersion is calculated for each firm according to equation (27).

- **Idiosyncratic Volatility**: Idiosyncratic volatility is calculated for each firm according to equation (29).

- **Credit Risk**: We examine credit risk by estimating the probability of default. To estimate the default probability, we calculate firm value at maturity, given by

$$P_T^i = D_T^i \int_T^\infty S(X_T, \tau) \left[ e^{-\alpha \tau} + \frac{\bar{\theta}}{\theta_T} (1 - e^{-\alpha \tau}) \right] ds.$$
We calculate the percentage of simulations for which \( P_T^i < B \), where \( B \) is the face value of debt, to estimate the probability of default.

- **Expected Return**: For each firm, we calculate expected firm returns (equivalent to the expected return on unlevered equity given in equation (22)) and simulated expected levered equity returns. To estimate expected levered equity returns, we calculate the realized return on levered equity in each of the 10,000 simulations and average these realizations. The realized return on levered equity in each simulation is \( R^i = \frac{V_T^i + \int_0^T D_t^idt}{V_0^i} \), where \( V_T^i = \max(P_T^i - B, 0) \) and \( \int_0^T D_t^idt \) is calculated given the sequences for \( D_t \) and \( \theta_t \). To estimate \( V_0^i \), we find

\[
V_0^i = E_t \int_0^T \frac{\Lambda_t}{\Lambda_0} D_t^idt + E_t \left[ \frac{\Lambda_T}{\Lambda_0} \max(P_T^i - B, 0) \right]
\]

given the sequences of \( \Lambda_t, D_t, \theta_t, \) and \( X_t \). The realized returns are averaged across simulations to estimate expected levered equity returns.

- **Beta**: For each firm, we calculate the unlevered beta and simulate the levered equity beta. Unlevered beta is calculated according to equation (24). Levered beta is simulated by dividing the estimated expected return on levered equity by the market expected return.

## C Aggregate Parameter Estimation

Our analysis requires the estimation of aggregate parameters which we borrow directly from Hore (2008). Based on post-war (1948-2006) data on aggregate dividends from the CRSP Value-Weighted Market Portfolio and aggregate real per-capita consumption
growth from BLS (both at annual frequency), posterior estimates of aggregate parameters \( \{\sigma_C, \sigma_D, \sigma_X, \mu_D, \mu_C, \lambda, \kappa\} \) are derived using a Bayesian MCMC mechanism. The most significant parameter is \( \lambda < 1 \) which is quite informative in the data. The resulting parameters can fit aggregate consumption and dividend growth rates quite well. Following are the median draws of the posterior distribution of the parameters: \( \sigma_C = 0.01, \sigma_D = 0.06, \sigma_X = 0.02, \mu_D = 0.05, \mu_C = 0.01, \lambda = 0.23, \kappa = 0.05 \).
References


Hore, Satadru, 2008, Equilibrium predictability and other asset pricing characteristics, Working paper, University of Iowa.


Wright, Sewall, 1931, Evolution in Mendelian populations, *Genetics* 16, 97–159.


Table 1: Correlations from Model Simulation

We simulate our model with a cross section of 500 firms and calculate the cross-sectional correlations of the model estimates of expected returns of levered equity, dispersion, idiosyncratic volatility, and default probability. Dispersion and idiosyncratic volatility are calculated according to equations (27) and (29), respectively. We estimate the probability of default by finding the proportion of the 10,000 iterations of the simulation in which the firm defaults on its debt obligations in the Merton (1974) framework. The expected return of levered equity is estimated by averaging realized levered equity returns across the 10,000 iterations. A complete description of the simulation procedure is included in Appendix (B). Panel A reports the correlation matrix for the entire cross section of firms. Panels B-D report correlation matrices within subsets of the cross section formed by sorting firms into terciles based on market value.

<table>
<thead>
<tr>
<th></th>
<th>Expected Return</th>
<th>Dispersion</th>
<th>Idiosyncratic Volatility</th>
<th>Default Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: All Firms</td>
<td>1.00</td>
<td>-0.61</td>
<td>-0.63</td>
<td>-0.51</td>
</tr>
<tr>
<td>Expected Return</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dispersion</td>
<td>-0.61</td>
<td>1.00</td>
<td>0.73</td>
<td>0.54</td>
</tr>
<tr>
<td>Idiosyncratic Volatility</td>
<td>-0.63</td>
<td>0.73</td>
<td>1.00</td>
<td>0.83</td>
</tr>
<tr>
<td>Default Probability</td>
<td>-0.51</td>
<td>0.54</td>
<td>0.83</td>
<td>1.00</td>
</tr>
<tr>
<td>Panel B: Small Firms</td>
<td>1.00</td>
<td>-0.41</td>
<td>-0.55</td>
<td>-0.54</td>
</tr>
<tr>
<td>Expected Return</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dispersion</td>
<td>-0.41</td>
<td>1.00</td>
<td>0.60</td>
<td>0.44</td>
</tr>
<tr>
<td>Idiosyncratic Volatility</td>
<td>-0.55</td>
<td>0.60</td>
<td>1.00</td>
<td>0.87</td>
</tr>
<tr>
<td>Default Probability</td>
<td>-0.54</td>
<td>0.44</td>
<td>0.87</td>
<td>1.00</td>
</tr>
<tr>
<td>Panel C: Medium Firms</td>
<td>1.00</td>
<td>-0.65</td>
<td>-0.54</td>
<td>-0.33</td>
</tr>
<tr>
<td>Expected Return</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dispersion</td>
<td>-0.65</td>
<td>1.00</td>
<td>0.65</td>
<td>0.44</td>
</tr>
<tr>
<td>Idiosyncratic Volatility</td>
<td>-0.54</td>
<td>0.65</td>
<td>1.00</td>
<td>0.77</td>
</tr>
<tr>
<td>Default Probability</td>
<td>-0.33</td>
<td>0.44</td>
<td>0.77</td>
<td>1.00</td>
</tr>
<tr>
<td>Panel D: Large Firms</td>
<td>1.00</td>
<td>-0.74</td>
<td>-0.60</td>
<td>-0.45</td>
</tr>
<tr>
<td>Expected Return</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dispersion</td>
<td>-0.74</td>
<td>1.00</td>
<td>0.63</td>
<td>0.29</td>
</tr>
<tr>
<td>Idiosyncratic Volatility</td>
<td>-0.60</td>
<td>0.63</td>
<td>1.00</td>
<td>0.63</td>
</tr>
<tr>
<td>Default Probability</td>
<td>-0.45</td>
<td>0.29</td>
<td>0.63</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Table 2: Monthly Investment Strategy Returns from Model Simulation

We simulate our model with a cross section of 500 returns and examine investment strategies based on estimated dispersion, idiosyncratic volatility, and probability of default. We sort the firms in the cross section of our simulation into deciles based on dispersion, idiosyncratic volatility, and credit risk. We form equally-weighted portfolios based on these sorts. The Low and High columns report the monthly expected return for the low and high extreme deciles of dispersion, idiosyncratic volatility, and credit risk. The Low-High columns report monthly expected returns from a strategy that buys the Low portfolio and sells the High portfolio. The returns for All Firms portfolios are based on the entire cross section of simulated firms. The Small Firms, Medium Firms, and Large Firms portfolios were formed by first sorting into terciles based on market value and then sorting based on dispersion, idiosyncratic volatility, or credit risk. In Panels A and B, we report investment strategy expected returns for equity investments in both levered and unlevered firms. Dispersion and idiosyncratic volatility are calculated according to equations (27) and (29), respectively. We estimate the probability of default by finding the proportion of the 10,000 iterations of the simulation in which the firm defaults on its debt obligations in the Merton (1974) framework. The expected return of levered equity is estimated by averaging realized levered equity returns across the 10,000 iterations. A complete description of the simulation procedure is included in Appendix (B).

<table>
<thead>
<tr>
<th>Panel A: Dispersion</th>
<th>Low</th>
<th>High</th>
<th>Low-High</th>
<th>Panel B: Idiosyncratic Volatility</th>
<th>Low</th>
<th>High</th>
<th>Low-High</th>
<th>Panel C: Credit Risk</th>
<th>Low</th>
<th>High</th>
<th>Low-High</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Levered Equity</td>
<td></td>
<td></td>
<td></td>
<td>Levered Equity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>All Firms</td>
<td>1.75</td>
<td>1.30</td>
<td>0.45</td>
<td>Unlevered Equity</td>
<td>0.80</td>
<td>0.61</td>
<td>0.19</td>
<td>All Firms</td>
<td>1.64</td>
<td>1.26</td>
<td>0.38</td>
</tr>
<tr>
<td>Small Firms</td>
<td>1.70</td>
<td>1.28</td>
<td>0.42</td>
<td>0.80</td>
<td>0.77</td>
<td>0.60</td>
<td>0.17</td>
<td>Small Firms</td>
<td>1.65</td>
<td>1.09</td>
<td>0.56</td>
</tr>
<tr>
<td>Medium Firms</td>
<td>1.74</td>
<td>1.33</td>
<td>0.41</td>
<td>0.79</td>
<td>0.76</td>
<td>0.62</td>
<td>0.14</td>
<td>Medium Firms</td>
<td>1.61</td>
<td>1.32</td>
<td>0.29</td>
</tr>
<tr>
<td>Large Firms</td>
<td>1.77</td>
<td>1.37</td>
<td>0.40</td>
<td>0.81</td>
<td>0.79</td>
<td>0.62</td>
<td>0.17</td>
<td>Large Firms</td>
<td>1.65</td>
<td>1.43</td>
<td>0.22</td>
</tr>
</tbody>
</table>
Figure 1: **Graph of** \( S(X_t, \tau) \). The function \( S(X_t, \tau) \), given in equation (9), is graphed under median draws from the posterior distribution of aggregate parameters (see Appendix (C) for details on aggregate parameter estimation). As shown in equation (8), the function \( S(X_t, \tau) \) is integrated over positive \( \tau \) to find the aggregate price-dividend ratio. As such, this function shows the value that arises from dividends expected to be paid at different horizons \( \tau \). The solid line shows \( S(X_t, \tau) \) for \( X_t = 0 \), while the top (bottom) dotted line shows \( S(X_t, \tau) \) for \( X_t = \sigma X \sqrt{2\kappa} \) (\( X_t = -\frac{\sigma X}{\sqrt{2\kappa}} \)).

Figure 2: **Implied discount rate for expected future cash flows.** This graph shows the implied annual discount rate for expected future cash flows as a function of cash flow horizon. The implied discount rate is the interest rate at which expected cash flows must be discounted to arrive at their current value. As shown in the figure, investors discount short-run cash flows at low rates compared to long-run cash flows. The implied annual discount rate was calculated using expected aggregate dividends \( E_t[D_s] \) at a series of horizons \( \tau = s - t \) and the value of those dividends \( D_t S(X_t, \tau) \). The implied discount rate was calculated using median draws from the posterior distribution of aggregate parameters (see Appendix (C) for details on aggregate parameter estimation) and \( X_t = 0 \).
Figure 3: **Expected Return and Relative Share.** The annual expected excess return of firm \( i \) is

\[
\mu_i^t = \frac{\Lambda(\gamma - 1)}{\kappa + \beta} \sigma_X^2 \frac{G_i}{G_X} + \beta \sigma_X^2 X_t G_i X_G
\]

which is a function of the firm’s relative share \( \bar{\theta}_i/\theta_t \). The firm is assumed to have no leverage. Expected excess returns are calculated using median draws from the posterior distribution of aggregate parameters (see Appendix (C) for details on aggregate parameter estimation) and \( X_t = 0 \).

Figure 4: **Conditional Beta on Economic Growth Risk Factor.** The conditional beta of firm \( i \) on the aggregate economic growth risk factor is given by

\[
\beta_i^t = \frac{G_i^X}{G_X} \frac{G}{G_X}
\]

as shown in equation (24). The firm’s conditional beta is a function of its relative share \( \bar{\theta}_i/\theta_t \) and the aggregate economic state \( X_t \). The firm is assumed to have no leverage. Conditional betas are calculated using median draws from the posterior distribution of aggregate parameters (see Appendix (C) for details on aggregate parameter estimation).
Figure 5: **Expected Return on Levered Equity.** This graph depicts the relationship between estimated expected returns on levered equity and relative share. The expected return on levered equity is estimated through simulation from our economy with a cross section of 500 firms. Each firm has randomly chosen firm-level parameters and a debt ratio of 0.5. The estimated expected return is an average of the realized returns on firm equity across the 10,000 simulation iterations. The expected return on levered equity is plotted against relative share $\frac{\hat{\theta}}{\theta}$ for each firm in the cross section. Complete details on the simulation procedure are provided in Appendix (B). Firms with relative share greater than 10 are censored to improve the clarity of the graph, but follow the expected pattern.

Figure 6: **Levered and Unlevered Betas on Economic Growth Risk Factor.** Estimated levered and unlevered equity betas are plotted against firm relative share $\frac{\hat{\theta}}{\theta}$ for simulated firms. The levered equity betas are estimated for each of the 500 firms in the cross section of our model simulation. Each firm has randomly chosen firm-level parameters and a debt ratio of 0.5. Levered equity betas are estimated by dividing the expected return on levered equity by the expected return on the hedge portfolio. For comparison, we also plot unlevered equity betas for firms with identical firm-level parameters as the levered firms in our cross section. Unlevered equity betas are calculated using equation (24). Complete details on the simulation procedure are provided in Appendix (B). Firms with relative share greater than 10 are censored to improve the clarity of the graph, but follow the expected pattern.
Figure 7: **Dispersion and Relative Share.** Dispersion is a function of firm relative share $\frac{\bar{\theta}}{\theta_0}$. Dispersion is defined as the standard deviation of firm dividend growth divided by the absolute value of expected dividend growth, as shown in equation (27). We plot dispersion versus relative share for each of the 500 firms in the cross section of our model simulation. Each firm has randomly chosen firm-level parameters. Complete details on the simulation procedure are provided in Appendix (B). Firms with relative share greater than 10 are censored to improve the clarity of the graph, but follow the expected pattern.

Figure 8: **Dispersion and Expected Returns on Unlevered Equity.** This graph shows the relationship between dispersion and the expected return on unlevered equity. Dispersion is defined as the standard deviation of firm dividend growth divided by the absolute value of expected dividend growth, as shown in equation (27). The expected return on unlevered equity is given by equation (22). We plot dispersion versus the expected return on unlevered equity for each of the 500 firms in the cross section of our model simulation. Each firm has randomly chosen firm-level parameters. Firms are sorted into terciles based on firm value at time 0. A red $\times$ represents a large firm, a green dot is a medium-sized firm, and a blue $+$ is a small firm. Complete details on the simulation procedure are provided in Appendix (B).
Figure 9: Dispersion and Expected Returns on Levered Equity. This graph shows the relationship between dispersion and the estimated expected return on levered equity. Dispersion is defined as the standard deviation of firm dividend growth divided by the absolute value of expected dividend growth, as shown in equation (27). The expected return on levered equity is estimated by averaging realized returns to levered equity across the 10,000 simulation iterations. We plot dispersion versus the estimated expected return on levered equity for each of the 500 firms in the cross section of our model simulation. Each firm has randomly chosen firm-level parameters and a debt ratio of 0.5. Firms are sorted into terciles based on firm value at time 0. A red × represents a large firm, a green dot is a medium-sized firm, and a blue + is a small firm. Complete details on the simulation procedure are provided in Appendix (B).

Figure 10: Idiosyncratic Volatility and Relative Share. Idiosyncratic volatility is a function of firm relative share \( \frac{\bar{v}}{v_0} \). Idiosyncratic volatility is given in equation (29). We plot idiosyncratic volatility versus relative share for each of the 500 firms in the cross section of our model simulation. Each firm has randomly chosen firm-level parameters. Complete details on the simulation procedure are provided in Appendix (B). Firms with relative share greater than 10 are censored to improve the clarity of the graph, but follow the expected pattern.
**Figure 11: Idiosyncratic Volatility and Expected Returns on Unlevered Equity.** This graph shows the relationship between idiosyncratic volatility and the expected return on unlevered equity. Idiosyncratic volatility is given in equation (29). The expected return on unlevered equity is given by equation (22). We plot idiosyncratic volatility versus the expected return on unlevered equity for each of the 500 firms in the cross section of our model simulation. Each firm has randomly chosen firm-level parameters. Firms are sorted into terciles based on firm value at time 0. A red × represents a large firm, a green dot is a medium-sized firm, and a blue + is a small firm. Complete details on the simulation procedure are provided in Appendix (B).

**Figure 12: Idiosyncratic Volatility and Expected Returns on Levered Equity.** This graph shows the relationship between idiosyncratic volatility and the estimated expected return on levered equity. Idiosyncratic volatility is given in equation (29). The expected return on levered equity is estimated by averaging realized returns to levered equity across the 10,000 simulation iterations. We plot idiosyncratic volatility versus the estimated expected return on levered equity for each of the 500 firms in the cross section of our model simulation. Each firm has randomly chosen firm-level parameters and a debt ratio of 0.5. Firms are sorted into terciles based on firm value at time 0. A red × represents a large firm, a green dot is a medium-sized firm, and a blue + is a small firm. Complete details on the simulation procedure are provided in Appendix (B).
Figure 13: **Credit Risk and Relative Share.** This graph depicts the relationship between the probability of default and relative share $\frac{\bar{\theta}}{\theta}$ for a levered firm. The probability of default is estimated for each of the 500 firms in the cross section of our model simulation by calculating the percentage of simulation iterations in which the firm defaults. Each firm has randomly chosen firm-level parameters and a debt ratio of 0.5. Following the Merton (1974) framework, a firm is assumed to default if its firm value at debt maturity is less than the face value of its debt. We plot estimated default probability versus relative share for each of the 500 firms in the cross section. Complete details on the simulation procedure are provided in Appendix (B). Firms with relative share greater than 10 are censored to improve the clarity of the graph, but follow the expected pattern.

Figure 14: **Credit Risk and Expected Returns on Levered Equity.** This graph shows the relationship between credit risk and the estimated expected return on levered equity. The probability of default is estimated for each of the 500 firms in the cross section of our model simulation by calculating the percentage of simulation iterations in which the firm defaults. Each firm has randomly chosen firm-level parameters and a debt ratio of 0.5. Following the Merton (1974) framework, a firm is assumed to default if its firm value at debt maturity is less than the face value of its debt. The expected return on levered equity is estimated by averaging realized returns to levered equity across the 10,000 simulation iterations. We plot estimated default probability versus the estimated expected return on levered equity for each of the 500 firms in the cross section of our model simulation. Firms are sorted into terciles based on firm value at time 0. A red $\times$ represents a large firm, a green dot is a medium-sized firm, and a blue $+$ is a small firm. Complete details on the simulation procedure are provided in Appendix (B).