Why Does Bad News Increase Volatility and Interest Rate, and Decrease Optimism, Asset Prices and Leverage?

F. Albert Wang

Abstract

We view mortgage as a risky derivative of its underlying house collateral and combine no-arbitrage valuation with equilibrium valuation approaches to develop a dynamic model of leverage cycle and interest rate. This model provides a unified explanation to pro-cyclical optimism, asset prices and leverage, and counter-cyclical volatility and interest rate. In addition, the model shows that tightening funding margin in the mortgage securities market dampens optimism, asset prices and leverage, whereas it raises volatility and interest rate in the housing market. Such double leverage cycle leads to more volatile markets and severe leverage cycle, thus resulting in worse financial crises.

Keywords: leverage; collateral; optimism, funding margin; interest rate, financial crisis; systemic risk, housing market; asset pricing

JEL Classification: E32; E44; G01; G12; R31

First draft: March 27, 2012

This Version: November 1, 2013

Please address correspondence to Department of Economics and Finance, University of Dayton, 300 College Park, Dayton, OH 45469-2251. Tel: (937) 229-3095; Fax: (937) 229-2477; E-mail: fwang1@udayton.edu. I thank Pete Kyle for valuable comments.

Why Does Bad News Increase Volatility and Interest Rate, and Decrease Optimism, Asset Prices and Leverage?

I. Introduction

The recent financial crisis of 2007-2009, the so called "Great Recession," was initially triggered by the collapse of the subprime mortgage in the housing market and ultimately caused much greater and wider damage to the American households than did the previous financial crisis triggered by the burst of the internet bubble during the turn of the millennium. Geanakoplos (2010b) posits that the recent financial crisis is the tail end of the recurrent "leverage cycle" phenomenon in American financial history. In an extreme manifestation of the leverage cycle, asset prices and leverage move together in huge swings and reinforce each other in a feedback loop until the market crashes. Indeed, the aftermath of the recent financial crisis has kindled growing research interest in the role of leverage on systemic risk in the economy.[1]

While most early economists focus on the role of interest rate in macroeconomic policies, Geanakoplos (1997, 2003) among other pioneers recognized early on the importance of leverage as a distinct equilibrium variable from the interest rate, as well as that the huge variations in leverage is a serious systemic risk to the economy.[2] Geanakoplos and others further develop the theory and applications of leverage cycle in which leverage is endogenously determined in a collateral equilibrium.[3] But, in this body of work, they often adopt risk-free mortgage, the so called "maxmin contract," in their analysis. This risk-free mortgage assumption obviously runs counter to the fact that the widespread default in the mortgage market is at the center of the recent financial crisis. More importantly, in this setup, while they are able to generate the endogenous leverage cycle dynamics, the equally important interest rate dynamics is void by construction. As evident in the recent financial crisis, we observed that both volatility and interest rate shot up while leverage tanked.

[1] See, for example, Brunnermeier and Pedersen (2009), Adrian and Shin (2010), Acharya and Viswanathan (2011), Garleanu and Pedersen (2011), Araujo et al. (2012), He and Xiong (2012), and Simsek (2013).

[2] Other notable early work on leverage or collateral includes, for example, Shleifer and Vishny (1992), Holmstrom and Tirole (1997), Kiyotaki and Moore (1997), Bernanke et al. (1999), Caballero and Krishnamurthy (2001), and Gromb and Vayanos (2002).

[3] See, for example, Geanakoplos and Zame (1997), Geanakoplos (2010a), and Fostel and Geanakoplos (2008a, 2008b, 2012a, 2012b).

In this paper, we provide a theoretical framework in which both the leverage cycle dynamics and the interest rate dynamics are endogenously determined in equilibrium. To do so, we treat mortgage as a risky derivative of its house collateral since both assets are subject to the same underlying fundamental and default risk. In particular, when homeowners default on their mortgage payments, the lender seizes their house collateral for the recovery value of the mortgage. In this setup, we are able to price both the derivative asset (i.e., mortgage) and the underlying asset (i.e., house collateral) under the law of one price or the no-arbitrage condition. Given the two asset prices, the endogenous leverage cycle dynamics and the interest rate dynamics can be evaluated simply as functions of these two endogenous price variables. To see this, let the principal of the mortgage be normalized to unity (one) and the prices of the mortgage and its collateral be denoted by q and p, respectively. The collateral rate is simply p/q; the margin (or haircut) is $1-q/p$; the leverage ratio is $p/(p-q)$; the risky interest rate is $1/q-1$. Thus, asset prices, collateral rate, margin, leverage, and interest rate are all intrinsically connected in our model as they are in the real-world market.

Another important innovation of our theoretical framework is to combine the no-arbitrage valuation approach of asset pricing with the equilibrium valuation approach of natural buyers employed in Geanakoplos (2003, 2010a). One advantage of the equilibrium valuation approach is that agents' optimal consumption plans and financing strategies can be endogenously determined in the model. In our model, the "marginal belief" of agents under the equilibrium valuation approach is precisely the risk-neutral probability under the no-arbitrage valuation approach. Thus, we are able to use this pivotal connection to simultaneously determine asset prices under no arbitrage as well as agents' optimal consumption plans and financing strategies in a unique collateral equilibrium. To our knowledge, we are the first to combine these two valuation approaches to obtain such a dynamic model of leverage cycle and interest rate.

We then apply our model to re-examine the central question raised in Fostel and Geanakoplos (2012b): why does bad news increase volatility and decrease leverage? To do so, following their example, we impose two restricted assumptions—risk-free mortgage and extreme payoff structure—to our model and thereby compare the "extreme bad volatility" (EBV) project, in which payoffs are only volatile in bad times, to the "extreme good volatility" (EGV) project, in which payoffs are only volatile in good times. In this restricted model, we effectively replicate their findings and conclusion that agents prefer the EBV project over the EGV project because the EBV project offers them higher initial price and leverage in normal times. This suggests that our model subsumes theirs as a special case of our model under risk-free mortgage. But, like in Fostel and Geanakoplos (2012b), the two restricted assumptions also cause counterfactual extreme outcomes, including flat interest rate dynamics and infinite leverage.

We therefore relax the two restricted assumptions and consider a more general payoff structure such that both projects are volatile in both times. This general payoff structure is able to capture the salient feature of the two kinds of projects in Fostel and Geanakoplos (2012b) without suffering from the extreme payoff problem. We then evaluate these two general (BV and GV) projects in our model now under risky mortgage. Our model shows that while the BV project still offers higher initial leverage, but it no longer offers higher initial price. Nonetheless, as long as agents prefer higher initial leverage, they will still select the BV project for their investment. Our model shows that the BV project generates pro-cyclical optimism, asset prices and leverage, and counter-cyclical volatility and interest rate. Thus, our model under risky mortgage is able to provide a robust and unified explanation to the phenomenon that bad news increases volatility and interest rate, and decrease optimism, asset prices and leverage—as clearly evident in the recent financial crisis.

Lastly, we add a secondary leverage cycle in the mortgage securities market to our model of the primary leverage cycle in the housing market. The extended model captures the "double leverage cycle" noted in Geanakoplos (2010a, 2010b). We find that all our findings under single leverage cycle remain intact in the extended model with double leverage cycle. In addition, we find that tightening the funding margin in the secondary leverage cycle dampens optimism, asset prices and leverage, whereas it raises volatility and interest rate. Moreover, such tightening policy magnifies the leverage cycle as the market condition changes. Hence, double leverage cycle generates more volatile markets and severe leverage cycle, and worse financial crises.

The contribution of our paper to the literature is fourfold. First, we are the first to combine no-arbitrage and equilibrium valuation approaches to obtain a dynamic model of leverage cycle and interest rate. Second, we extend the original leverage cycle model of Geanakoplos (2003, 2010a) under risk-free mortgage to a general model under risky mortgage. Third, we strengthen the main findings in Fostel and Geanakoplos (2012b) and provide a robust and unified explanation to pro-cyclical optimism, asset prices, and leverage, and counter-cyclical volatility and interest rate. Fourth, our extended model with double leverage cycle yields new testable implications concerning the marginal effect of funding margin in the secondary cycle on the primary cycle of the housing market. In addition, such double leverage cycle leads to more severe leverage cycle and worse financial crises.

The remainder of the paper is organized as follows. In section II, we introduce a one-shot model to illustrate the basic setup of our theoretical framework. In section III, we extend the one-shot model into a dynamic model, which generalizes the collateral equilibrium of Geanakoplos (2003, 2010a). In section IV, following Fostel and Geanakoplos (2012b), we compare the EBV project to the EGV project in a restricted model. In section V, we compare the BV project to the GV project in our general model. In section VI, we extend our model to incorporate double leverage cycle. In Section VII, we conclude. All proofs are in the Appendix.

II. The One-Shot Model

A. No Arbitrage Valuation: The Asset Prices under the Law of One Price

Following Wang and Zhang (2013), consider a one-shot model from t=0 to t=1 with two states of nature at t=1: good state (s=U) and bad state (s=D). Of course, there is only one state of nature (s=0) at the beginning of the period at t=0. Consumption good (X) and three assets (Y, Z, and B) are traded in the one shot market among agents with heterogeneous beliefs. X is consumption good and also serves as numeraire with a fixed price of 1 in all states. Y is an investment good (i.e., the house) that delivers u units of X in the good state (s=U) and d units of X in the bad state (s=D) such that $d < 1 < u$. Z is a financial contract (i.e., the mortgage loan) that uses the investment good Y as its collateral and promises to pay 1 unit of X as principal, regardless which state of nature is realized at t=1.[4] The lender knows, however, that the borrower will only honor the obligation in the good state (s=U), and default in the bad state (s=D). In the latter state, the lender seizes the collateral and recovers d units of X from it at t=1. Thus, we allow the risky mortgage Z to default in bad times. This is a sharp departure from the model of risk-free mortgages in Geanakoplos (2003, 2010a). Moreover, the risky mortgage Z is in effect a derivative of its collateral Y. That is, the derivative Z pays 1 unit of X in s=U and d units of X in

[4] We later relax the exogenous principal of 1 unit and let it be contingent on the current market price of the collateral in the dynamic model.

s=D, depending on the outcome of its collateral Y. B is a risk-free bond that pays a fixed amount of 1 unit of X regardless of which state is realized at t=1.

Since the payoff of each of the three assets (Y, Z, and B) can be replicated by some combination of the other two assets, the law of one price dictates that all three assets are uniquely priced by a risk-neutral probability, denoted by *a*, to prevent arbitrage. Given that the risk-free bond B pays 1 unit of X at t=1, its price is simply given by $1/(1+r_f)$, where r_f is the risk-free rate. Let the price of the investment good Y be denoted by $p = p(Y)$ and the price of its derivative Z be denoted by $q = q(Z)$. No arbitrage condition requires that the holding-period return of the risky investment good in the good state (s=U) at t=1 is greater than the risk-free holding-period return, $1+r_f$, which in turn is greater than the return in the bad state (s=U), i.e., $d/p < 1+r_f < u/p$. This condition ensures that the risk-neutral probability *a* is an interior solution such that $0 < a < 1$. Given the risk-neutral probability *a*, the prices of the investment good Y, and the mortgage Z, respectively, are given by

$$p = p(Y) = (a \cdot u + (1-a)d)/(1+r_f) \quad (1)$$

$$q = q(Z) = (a \cdot 1 + (1-a)d)/(1+r_f) \quad (2)$$

B. Equilibrium Valuation: The Housing Market with Heterogeneous Natural Buyers

Following Geanakoplos (2003, 2010a), there are a continuum of risk-neutral agents $h \in H = [0,1]$ with heterogeneous beliefs about the state of nature at t=1 such that agent *h* believes that the good state (s=U) occurs with probability *h* while the bad state (s=D) occurs with probability 1-*h*.[5] Given agent *h*'s subjective belief, his/her expected present value of the investment good Y, denoted by $p^h(Y)$, is

$$p^h(Y) = (h \cdot u + (1-h)d)/(1+r_f) \quad (3)$$

Clearly, the subjective value of the house, $p^h(Y)$, increases in $h \in H = [0,1]$ across agents, although the market price of the house is uniquely determined based on the risk-neutral

[5] As Geanakoplos (2003, 2010) points out, the heterogeneity in beliefs in this kind of setup may be regarded as a reduced-form version of a comparable model with risk-averse agents such that low-*h* agents are more risk averse.

probability *a* according to Eq (1). Thus, the agents with high *h*'s (i.e., $a < h \le 1$) will consider the investment good Y "underpriced" whereas the agents with low *h*'s (i.e., $0 \le h < a$) will consider it "overpriced." In this sense, the agents with high *h*'s are "optimists" and the agents with low *h*'s are "pessimists." The agent whose subjective belief coincides with the risk-neutral probability (i.e., *h*=*a*) is the "marginal agent" who thinks the investment good Y is neither "underpriced" nor "overpriced." As such, the risk-neutral probability "*a*" under the no-arbitrage framework is precisely the marginal belief of the natural buyers under the equilibrium framework.

Following Geanakoplos (2003, 2010a), assume that agents care only about their consumption of dollars in units of the consumption good X. In addition, we assume that agents are impatient and discount future consumption of X. Let x_0^h, x_U^h, x_D^h be the units of X consumed in state $s = 0, U, D$, respectively. The expected utility of agent *h*, denoted by U^h, is

$$U^h(x_0^h, x_U^h, x_D^h) = x_0^h + (h \cdot x_U^h + (1-h) \cdot x_D^h) / (1 + r_f) \qquad (4)$$

Suppose further that all agents are endowed with *e* units of consumption good X and 1 unit of the investment good Y in state s=0 and nothing otherwise. Given the endowment, each agent maximizes his/her expected utility by making optimal investment as well as consumption decision at t=0, which affects his/her consumption plan at t=1.

Consider first the "no-trade" case in which each agent simply consumes his endowment *e* at t=0 and consumes *u* units of X in good state (s=U) or *d* units of X in bad state (s=D). In this case, the utility increases in $h \in H = [0,1]$ across agents such that

$$U^{h<a}(e,u,d) = e + p^{h<a} < U^{h=a}(e,u,d) = e + p < U^{h>a}(e,u,d) = e + p^{h>a} \qquad (5)$$

Given the market price *p* for the investment good Y, the inequality in Eq (5) implies that the pessimist ($0 \le h < a$) can increase his/her utility by selling the investment good Y, whereas the optimist ($a < h \le 1$) can increase his/her utility by buying the investment good Y. Thus, the no-trade result is not an equilibrium and there will be utility gains from trading of the investment good Y between the pessimists and the optimists. In particular, it is optimal that the pessimists sell all their endowments of the investment good Y to the optimists at the market price *p* per unit of Y at t=0. Furthermore, given the linear utility function with the discount rate r_f, the pessimists will choose to consume all the consumption good X they can obtain now at t=0, whereas the optimists will choose to use their endowments of X to purchase Y from the pessimists at t=0 and to consume later at t=1 from the residual claim of their investment good Y.

Moreover, the optimist can leverage their purchase of the investment good Y by borrowing q with the mortgage Z against Y as its collateral. That is, the buyers of Y only need to give $p-q$ per unit of Y (the house) as the down payment, while financing the rest with the mortgage Z. In so doing, the buyers maximize their purchasing power via such leverage.

In this market, all agents maximize their utility by choosing their optimal consumption plans along with optimal financing (borrowing or lending) strategies. This optimization problem can be formalized as follows. At t=0, given the price of the house p and that of the mortgage q, each agent h maximizes his/her expected utility by choosing the quantity of X, denoted by x_0^h, to consume at t=0, the quantity of contracts, denoted by φ^h, of the mortgage Z to borrow, and the quantity of contracts, denoted by θ^h, of the mortgage Z to lend.

Given the linear utility function in Eq (4), the marginal agent $a \in (0,1)$ will be indifferent to buying and selling Y at t=0, since he/she thinks the investment good Y is neither "underpriced" nor "overpriced." As we conjectured above, the optimistic agents $h \in (a,1]$ will buy all they can afford of Y by selling all their X and borrowing the mortgage Z to the maximum; the pessimistic agents $h \in [0,a)$ will sell all they have of Y, consume all endowments X at t=0, and lend via the mortgage Z to borrowers. Thus, the total sale of Y is a units times the price p per unit, resulting in a total payment of $a \cdot p$. The payment is made by all optimistic agents $h \in (a,1]$ from their X endowments, totaling $(1-a)e$, and their mortgage loan, totaling q. Thus, the market clearing condition for the housing market is given by

$$(1-a)e + q = a \cdot p \tag{6}$$

C. The Equilibrium

Given the exogenous variables e, r_f, u and d, all three endogenous variables (a, p, q) can be jointly determined by solving Eqs (1), (2) and (6) simultaneously under the constraint that the marginal belief a is a proper probability measure such that $0 < a < 1$. By doing so, we obtain a unique solution of the three endogenous variables as follows.

$$a = \frac{1-2d-e(1+r_f)+\sqrt{(1-2d-e(1+r_f))^2+4(d+e(1+r_f))(u-d)}}{2(u-d)}$$
$$p = \frac{1-e(1+r_f)+\sqrt{(1-e(1+r_f))^2+4d(u-1)+4u\cdot e(1+r_f)}}{2(1+r_f)} \tag{7}$$
$$q = \frac{(1-d)\left(1-e(1+r_f)+\sqrt{(1-e(1+r_f))^2+4d(u-1)+4u\cdot e(1+r_f)}\right)+2d(u-1)}{2(1+r_f)(u-d)}$$

This unique solution thus renders a unique collateral equilibrium (a,p,q) of the one-shot model as shown in Theorem 1 below.

Theorem 1. Given constants e, r_f, u and d, there exists a unique equilibrium (a, p, q) as given in Eq (7). *The resulting optimal financing strategies* (θ^h,φ^h) *and optimal consumption plans* (x_0^h,x_U^h,x_D^h) *are given as follows:*

(i) For each "pessimistic" agent $h\in[0,a)$,

(a) the quantities of mortgage contracts lent and borrowed are: $\theta^h=\frac{1}{a}$ *and* $\varphi^h=0$.

(b) the quantities of consumption good are: $x_0^h=\frac{1}{a}e$, $x_U^h=\frac{1}{a}$, and $x_D^h=\frac{1}{a}d$.

(ii) For each "optimistic" agent $h\in(a,1]$,

(a) the quantities of mortgage contracts lent and borrowed are: $\theta^h=0$ *and* $\varphi^h=\frac{1}{1-a}$.

(b) the quantities of consumption good are: $x_0^h=0$, $x_U^h=\frac{1}{1-a}(u-1)$, *and* $x_D^h=0$.

Theorem 1 shows that at t=0, the pessimists consume all consumption good X, sell all their investment good Y, and lend mortgage Z, whereas the optimists consume none, own all investment good Y, and borrow mortgage Z. At t=1, both groups have their contingent consumption plans, depending on which state, s=U or s=D, occurs at that time.

The one-shot model is useful to illustrate the crux of our analysis in which the marginal belief and the two asset prices affect each other and are jointly determined in the collateral equilibrium. To examine the dynamics in a multi-period setup, one needs to extend the one-shot model into a dynamic one, which is the subject we turn to in what follows.

III. The Dynamic Model

The one-shot model can be extended into a multi-period one in which trading takes place at the beginning of each period, and the ultimate outcome depends critically on what happens in the interim periods. For our purposes, it is sufficient to study the essential dynamics in a two-period setting that allows the house price, *p*, to rise or fall in the interim period before its ultimate fundamental value is revealed at the end of the second period. As Geanakoplos (2003, 2010a) points out, the housing market can crash in the interim period even before the fundamental value is to be realized at the end of the second period. Therefore, we focus our analysis on how the dynamics between asset prices and leverage in the interim period affect the ultimate outcomes in the mortgage (derivative asset) market and its housing collateral (underlying asset) market.

A. The Natural Buyers with Heterogeneous Beliefs

Extending the basic setup in the one-shot model, let the underlying state of nature follow the standard binomial asset pricing framework such that from the current state (s=0) at t=0, the nature can move to s=U or s=D at t=1, and move further to s=UU or s=UD at t=2 after s=U or to s=DU or s=DD at t=2 after s=D. We follow the standard assumption that each state is distinctly determined by the numbers of Us and Ds leading to it, but not path-dependent. Thus, without loss of generality, we can characterize the distinctive state of nature in the two-period setting by $s \in S = \{0, U, D, UU, UD, DD\}$.

As in the one-shot model, there is a continuum of risk-neutral agents $h \in H = [0,1]$ with heterogeneous beliefs about the state of nature such that agent *h* believes that the good state (s=U) occurs with probability *h* while the bad state (s=D) occurs with probability 1-*h* at each time when nature moves (i.e., at t=1 the first time and again at t=2 the second time). The resulting state of nature tree for agent *h* is thus depicted in Figure 1.

B. The Contingent House Prices

In this two-period model, trading of the house Y takes place first at time t=0 and state s=0 and subsequently at time t=1 in either stat s=U or state s=D, depending on which state is realized at t=1. The fundamental value of Y is realized at t=2 by its terminal payoff at that time. Denote by p_s the price of Y in state s for $s \in S = \{0, U, D, UU, UD, DD\}$. Without loss of generality, let the terminal payoff at t=2 be exogenously given to capture the underlying fundamental in each state such that $p_{UU} = u$, $p_{UD} = v$, and $p_{DD} = d$, where $d < v < u$. In contrast, the beginning house price p_0 in state s=0 as well as the contingent house prices p_U and p_D in states s=U and s=D are endogenously determined in the market.

As in the one-shot model, the market price of the house in each of the three initial states (i.e., s=0, s=U, s=D) is uniquely determined by its corresponding risk-neutral probability, which is also the belief of the marginal agent in each state. Let a_0, a_U, a_D be the marginal beliefs in state s=0, s=U, and s=D, respectively. The market prices p_0, p_U, and p_D of the house Y in these states are therefore given by

$$p_0 = (a_0 \cdot p_U + (1 - a_0) p_D) / (1 + r_f) \quad (8)$$

$$p_U = (a_U \cdot u + (1 - a_U) v) / (1 + r_f) \quad (9)$$

$$p_D = (a_D \cdot v + (1 - a_D) d) / (1 + r_f) \quad (10)$$

The resulting house price tree, $p_s \in P = \{p_0, p_U, p_D, p_{UU}, p_{UD}, p_{DD}\}$, is thus depicted in Figure 2. Of course, the focus of our analysis will be on the determination of the endogenous market prices (p_0, p_U, p_D) and how they interact with the endogenous leverage cycle in the equilibrium.

C. The Contingent Mortgage Prices

As in the one-shot model, a house buyer can obtain a mortgage on the house to leverage his/her purchase. One of the key debates on the cause of the recent financial crisis centers on the misalignment of maturities between the long-term asset and the short-term debt used in the leveraged purchase. To capture such a maturity misalignment in our model, consider the set of one-period mortgages $Z = (Z_0, Z_U, Z_D)$ with its corresponding prices $q = (q_0, q_U, q_D)$ such that the short-term mortgage Z_s with its price q_s is available in state s for $s \in \{0, U, D\}$.

Furthermore, we relax the assumption of the exogenous principal in the one-shot model by allowing the principal amount to be contingent on the current house price, p_s. This is to capture the "real-world" practice that the principal of the mortgage typically depends on its underlying collateral value.

As in the one-shot model, even though the borrowers of the short-term mortgage promise to pay the principal regardless of what happens in the next period, the lenders know, however, that the borrowers will only honor the obligation in the good state and default in the bad state. In the former case, the borrowers (i.e., the optimistic agents) pay the mortgage and keep their houses. In the latter state, the lenders (i.e., the pessimistic agents) seize the collateral and recover the remaining value of the house. When this happens, the optimistic agents are effectively driven out of the market and leave all their houses to the pessimistic agents.

In this setup, each mortgage Z_s available in state s for $s \in \{0, U, D\}$ is in effect a derivative of its underlying collateral (the house). Specifically, the price of the mortgage Z_0, which is q_0 in s=0, can either rise to p_0 if s=U occurs or fall to p_D if s=D occurs at t=1. Likewise, the price of the mortgage Z_U, which is q_U in s=U, can either rise to p_U if s=UU occurs or fall to *v* if s=UD occurs at t=2. And, the price of the mortgage Z_D, which is q_D in s=D, can either rise to p_D if s=UD occurs or fall to *d* if s=DD occurs at t=2.

Since each contingent mortgage Z_s is a derivative of its underlying collateral Y in state *s* for $s \in \{0, U, D\}$, its price, q_s, is therefore also determined by its corresponding risk-neutral probability at each state, a_s, as follows:

$$q_0 = (a_0 \cdot p_0 + (1 - a_0) p_D) / (1 + r_f) \quad (11)$$

$$q_U = (a_U \cdot p_U + (1 - a_U) v) / (1 + r_f) \quad (12)$$

$$q_D = (a_D \cdot p_D + (1 - a_D) d) / (1 + r_f) \quad (13)$$

The three contingent mortgage price trees, $q_s \in Q = \{q_0, q_U, q_D\}$, are thus depicted in Figure 3.

D. The Housing Market Clearing Conditions

As in the one-shot model, agents care only about their consumption of dollars, i.e., units of the consumption good X, and they are impatient and discount future consumption. It is worth noting that in the dynamic model, the population of agents in the housing market changes from *t*=0 to t=1, depending on which state, *s*=U or *s*=D, occurs at *t*=1. Initially, all agents are endowed with 1 unit of the house Y at time *t*=0. At time *t*=1, if state *s*=U occurs, the pessimistic agents' mortgages are paid in full and the optimistic agents (i.e., $h \in (a_0, 1]$ with a population of $1-a_0$ keep all stock of the houses in the market. If, however, state *s*=D occurs at *t*=1, the optimistic agents default on their mortgages and leave all their houses to the pessimistic agents (i.e., $h \in [0, a_0)$) with a population of a_0.

Suppose the aggregate endowment of consumption good X is a constant *e* at the beginning of each period (i.e., *t*=0 and *t*=1), but none at the terminal time t=2. That is, denote by e_s the aggregate endowment of consumption good X in state *s*. The aggregate endowment tree is thus given by $\{e_0, e_U, e_D, e_{UU}, e_{UD}, e_{DD},\} = \{e, e, e, 0, 0, 0\}$.[6] Without loss of generality, let the aggregate endowment be equally distributed among the existing agents in the housing market at each state. Suppose further that each agent is also initially endowed with 1 unit of the investment good (the house) Y at t=0 and none in later periods. Thus, there is a constant stock of 1 unit of the house Y in the market at all times. At the beginning of each period (i.e., t=0 and t=1), the existing agents in the market can trade their houses among themselves at the prevailing market price to maximize their utility. In other words, the existing agents in the market will trade in each of the three initial states: *s*=0, *s*=U and *s*=D. In the remainder of this subsection, we describe the market clearing condition at each of these states. Following the standard backward induction analysis in a dynamic model, we consider the case of the state *s*=U first, then the case of the state *s*=D, and lastly the case of the state *s*=0.

D1. The Housing Market in State s=U

When the good state occurs in s=U, the price of the house Y rises from p_0 in s=0 to p_U. In this state, the optimistic agents (i.e., $h \in (a_0, 1]$) pay off their old mortgage Z_0 with a total

[6] We adopts Geanakoplos' (2003, 2010) assumption of a constant aggregate endowment of X in our analysis. For robustness, we also assume constant endowment of X per agent and the results are essentially the same.

(principal) payment of p_0 to the pessimistic agents. Recall that the new total endowment of the consumption good X in state s=U is *e*. Since the pessimistic agents take the mortgage payment p_0 from the optimistic agents as they leave the market, the net total endowment to the optimists as a whole in this state is $e - p_0$. The optimistic agents with a population of $1 - a_0$ trade once again among themselves to maximize their utilities in state s=U. Let $x_U^h, x_{UU}^h, x_{UD}^h$ be the units of X consumed in state $s = U, UU, UD$, respectively. Thus, the utility of each optimistic agent $h \in (a_0, 1]$ in the market is given by

$$U^h(x_U^h, x_{UU}^h, x_{UD}^h) = x_U^h + (h \cdot x_{UU}^h + (1-h) \cdot x_{UD}^h) / (1 + r_f) \qquad (14)$$

The optimization problem in s=U is a one-period problem which has the same form as the optimization problem analyzed in the one-shot model above, except that the exogenous, terminal payoffs of the house Y are different. Thus, as in the one-shot model, there will be gains from trading among the existing agents ($h \in (a_0, 1]$). And, each agent $h \in (a_0, 1]$ maximizes his/her expected utility by choosing the quantity of the consumption good (x_U^h) to consume and the quantity of the mortgage contract Z_U to lend (θ_U^h) or borrow (φ_U^h) in state s=U.

Note first that the market price of the house p_U in state s=U is uniquely determined in (9) according to the risk-neutral probability a_U, which is also the "marginal belief" in this state. Given the marginal belief a_U, those existing agents with a belief greater than the marginal belief are now the "new" optimistic agents (i.e., $h \in (a_U, 1]$) in state *s*=U, whereas those existing agents with a belief less than the marginal belief are the "new" pessimistic agents (i.e., $h \in (a_0, a_U)$). Similarly, those existing agents whose subjective belief coincides with the risk-neutral probability (i.e., *h*= a_U) are the "new" marginal agents.

Given the prevailing market price p_U, the "new" optimistic agents ($h \in (a_U, 1]$) will buy all they can still afford of Y by selling all their endowed X in s=U and borrowing the new mortgage loan Z_U to the maximum. The "new" pessimistic agents $(h \in (a_0, a_U))$ will sell all they have of Y, consume all endowed X in s=U, and lend the mortgage loan Z_U. The market clearing condition for the housing market in state s=U is therefore given by[7]

$$\frac{1-a_U}{1-a_0}(e-p_0)+q_U=\frac{a_U-a_0}{1-a_0}p_U \tag{15}$$

Given exogenous variables e, r_f, u and v, all three endogenous variables (a_U, p_U, q_U) can be jointly determined by solving Eqs (9), (12), and (15) simultaneously. As it turns out, the resulting marginal belief a_U is a unique, interior solution such that $a_U \in (a_0, 1)$.

D2. The Housing Market in State s=D

When the bad state occurs in s=D, the price of the house Y falls from p_0 to p_U. In this state, the market is populated with the pessimistic agents (i.e., $h \in [0, a_0)$) with a population of *a*, while the optimistic agents default on their mortgages and leave the market. Recall that the new total endowment of the consumption good X in the state s=D is *e*. Since the optimistic agents leave the market, the total endowment to the pessimists as a whole in the state is *e*. The pessimistic agents trade once again among themselves to maximize their utilities in state s=D at t=1. Let $x_D^h, x_{UD}^h, x_{DD}^h$ be the units of X consumed at $s = D, UD, DD$, respectively. The utility of each agent *h* in the market is thus given by

$$U^h(x_D^h, x_{UD}^h, x_{DD}^h) = x_D^h + (h \cdot x_{UD}^h + (1-h) \cdot x_{DD}^h)/(1+r_f) \tag{16}$$

---

[7] In s=U, the optimistic agents ($h \in (a_0, 1]$) hold all stock of Y with a population of $1-a_0$ and hence each agent owns $\frac{1}{1-a_0}$ units of Y. Among these existing agents, only the "new" pessimistic agents ($h \in (a_0, a_U)$) will sell all their Y, totaling $\frac{a_U-a_0}{1-a_0}$ units, at the prevailing price p_U per unit. As a result, the total payment of the sale is $\frac{a_U-a_0}{1-a_0}p_U$. Only the "new" optimistic agents ($h \in (a_U, 1]$) will use all their net endowment of X, totaling $\frac{1-a_U}{1-a_0}(e-p_0)$, as part of the payment to purchase Y. In addition, these "new" optimistic agents ($h \in (a_U, 1]$) use the entire stock of Y in the market (both their endowed Y and the purchased Y), hence a total value of q_U, to finance their purchase of Y. As a result, the total fund used to pay for the purchase of Y is $\frac{1-a_U}{1-a_0}(e-p_0)+q_U$.

This optimization problem in s=D is essentially the same one-period problem as that in s=U analyzed above and has the same form as the optimization problem analyzed in the one-shot model above, except that the exogenous terminal payoffs of the house Y are different. Thus, as in the one-shot model, there will be gains from trading among the existing agents ($h \in [0, a_0)$). And, each agent $h \in [0, a_0)$ maximizes his/her expected utility by choosing the quantity of the consumption good (x_D^h) to consume and the quantity of the mortgage contract Z_D to lend (θ_D^h) or borrow (φ_D^h) in state s=D.

Note here that the market price of the house p_D in state s=D is uniquely determined in (10) according to the risk-neutral probability a_D, which is also the "marginal belief" in this state. Given the marginal belief a_D, those existing agents with a belief greater than the marginal belief are now the "new" optimistic agents (i.e., $h \in (a_D, a_0)$) in state *s*=D whereas those existing agents with a belief less than the marginal belief are now the "new" pessimistic agents (i.e., $h \in [0, a_D)$). Similarly, those existing agents whose subjective belief coincides with the risk-neutral probability (i.e., $h = a_D$) are the "new" marginal agents.

Given the prevailing market price p_D, the "new" optimistic agents ($h \in (a_D, a_0)$) will buy all they can afford of Y by selling all their endowed X in state s=D and borrowing mortgage Z_D to the maximum. In contrast, the "new" pessimistic agents ($h \in [0, a_D)$) will sell all they have of Y, consume all endowment X in s=D, and lend mortgage Z_D. The market clearing condition for the housing market in s=D is therefore given by[8]

[8] In s=D, the pessimistic agents ($h \in [0, a_0)$) hold all stock of Y with a population of a_0 and hence each agent owns $\frac{1}{a_0}$ units of Y. Among these existing agents, only the "new" pessimistic agents ($h \in [0, a_D)$) will sell all their Y, totaling $\frac{a_D}{a_0}$ units, at the prevailing price p_D per unit. As a result, the total payment of the sale is $\frac{a_D}{a_0} p_D$. The aggregate endowment of X, which is *e*, is equally distributed among the existing agents with a population a_0. Thus, the endowment of X per agent is $\frac{1}{a_0} e$. Among the existing agents, only the "new" optimistic agents ($h \in (a_D, a_0)$) will use all their endowment of X, totaling $\frac{a_0 - a_D}{a_0} e$, as part of the payment to purchase Y. In addition, these "new"

$$\frac{a_0 - a_D}{a_0} e + q_D = \frac{a_D}{a_0} p_D \tag{17}$$

Given the exogenous variables e, r_f, v and d, all three endogenous variables (a_D, p_D, q_D) can be jointly determined by solving Eqs (10), (13), and (17) simultaneously. As it turns out, the resulting marginal belief a_D is a unique, interior solution such that $a_D \in (0, a_0)$.

D3. The Housing Market in State s=0

Following the backward induction analysis, we are now back to the initial period t=0 and state s=0. In this initial state, all agents (i.e., $h \in [0,1]$) participate in the market with the initial population of 1 and each agent is endowed with 1 unit of the investment good Y. These agents trade the first time to maximize their utilities in state s=0 while taking into account two possible outcomes s=U or s=D at t=1.

If s=U occurs, the optimistic agents (i.e., $h \in (a_0, 1]$) pay off the principal of the mortgage p_0 in full and own all stock of Y. In this case, although the optimistic agents do not consume immediately at t=1, their ownership of the investment good Y gives them the right to future consumption depending on the terminal payoff of Y at t=2. The present value at t=1 of such future consumption at t=2 is exactly the value of Y in state s=U, i.e., p_U. Since the agents only care about their consumption of dollars in units of X, the ownership of Y thus in effect gives the optimistic agents p_U quantities of consumption good X.

If s=D occurs, the pessimistic agents (i.e., $h \in [0, a_0)$) now own all stock of Y as the optimistic agents default on their mortgages. In this case, although the pessimistic agents do not consume immediately at t=1, their ownership of the investment good Y gives them the right to future consumption depending on the terminal payoff of Y at t=2. The present value at t=1 of such future consumption at t=2 is exactly the value of Y in state s=D, i.e., p_D. Since the agents

---

optimistic agents ($h \in (a_D, a_0)$) use the entire stock of Y in the market, hence a total value of q_D, to finance their purchase of Y. As a result, the total payment of the purchase is $\frac{a_0 - a_D}{a_0} e + q_D$.

only care about their consumption of dollars in units of X, the ownership of Y thus in effect gives the pessimistic agents p_D quantities of consumption good X.

In this context, the two contingent house prices p_U and p_D are part of the consumption plan that all agents at t=0 will consider to maximize their utilities. Let x_0^h, x_U^h, x_D^h be the units of X consumption good at $s = 0, U, D$, respectively. The utility of each agent (i.e., $U^h(x_0^h, x_U^h, x_D^h)$) in the market is exactly the same as that described in Eq (4) in the one-shot model above.

This optimization problem in state s=0 is thus essentially the same as that analyzed in the one-shot model above, except that the payoffs of the house Y at t=1 are not exogenously given, but rather are endogenously determined by the two contingent house prices p_U and p_D at t=1. Thus, as in the one-shot model, each agent $h \in [0,1]$ maximizes his/her expected utility by choosing the quantity of the consumption good (x_0^h) to consume and the quantity of the mortgage contract Z_0 to lend (θ_0^h) or borrow (φ_0^h) in state s=0.

Specifically, the optimistic agents $h \in (a_0, 1]$ will buy all they can afford of Y by selling all their X and borrowing the mortgage Z_0 to the maximum; the pessimistic agents $h \in [0, a_0)$ will sell all they have of Y, consume all endowments X at t=0, and lend via the mortgage Z_0 to borrowers. As a result, we obtain essentially the same market clearing condition as that in the one-shot model below:

$$(1 - a_0)e + q_0 = a_0 \cdot p_0 \qquad (18)$$

Given exogenous variables e, r_f, u, v and d, all three endogenous variables (a_0, p_0, q_0) can be jointly determined by solving Eqs (8), (11), and (18) simultaneously. As it turns out, the resulting marginal belief a_0 is indeed a unique, interior solution such that $a_0 \in (0,1)$.

E. The Equilibrium

Since both the solution $a_D \in (0, a_0)$ and the solution $a_U \in (a_0, 1)$ are functions of $a_0 \in (0,1)$, the uniqueness of the solution $a_0 \in (0,1)$ also ensures the uniqueness of a_D and a_U such that $0 < a_D < a_0 < a_U < 1$. Thus, starting from the initial marginal belief a_0 in state s=0, the marginal belief rises to a_U or falls to a_D as the market moves up or down in its corresponding states, s=U or s=D.

The uniqueness of the set of the marginal beliefs (a_0, a_U, a_D) also ensures a unique equilibrium of the dynamic model in which the three contingent house prices (p_0, p_U, p_D) and the three contingent mortgage prices (q_0, q_U, q_D) are also endogenously and uniquely determined as in Eqs (8)-(13). Theorem 2 reports the unique collateral equilibrium of the dynamic model below.

Theorem 2. Given constants e, r_f, u, v *and* d, *there exists a unique equilibrium (* a_0, a_U, a_D, $p_0, p_U, p_D, q_0, q_U, q_D$*) in which the optimal financing strategies* $(\theta_0^h, \theta_U^h, \theta_D^h, \varphi_0^h, \varphi_U^h, \varphi_D^h)$ *and the optimal consumption plans* $(x_0^h, x_U^h, x_D^h, x_{UU}^h, x_{UD}^h, x_{DD}^h)$ *in each state* $(s = 0, U, D)$ *are given as follows:*

*1. In state s=*0

(i) For each "pessimistic" agent $h \in [0, a_0)$,

(a) the quantities of mortgage contracts lent and borrowed are: $\theta_0^h = \dfrac{1}{a_0}$ *and* $\varphi_0^h = 0$.

(b) the quantities of consumption good are: $x_0^h = \dfrac{e}{a_0}$, $x_U^h = \dfrac{p_0}{a_0}$ *and* $x_D^h = \dfrac{p_D}{a_0}$.

(ii) For each "optimistic" agent $h \in (a_0, 1]$,

(a) the quantities of mortgage contracts lent and borrowed are: $\theta_0^h = 0$ *and* $\varphi_0^h = \dfrac{1}{1 - a_0}$.

(b) the quantities of consumption good are: $x_0^h = 0$, $x_U^h = \dfrac{p_U - p_0}{1 - a_0}$ *and* $x_D^h = 0$.

2. In state s=U

(i) For each "pessimistic" agent $h \in (a_0, a_U)$,

(a) the quantities of mortgage contracts lent and borrowed are: $\theta_U^h = \dfrac{1}{a_U - a_0}$ *and* $\varphi_U^h = 0$.

(b) the quantities of consumption good are: $x_U^h = \dfrac{e - p_0}{a_U - a_0}$, $x_{UU}^h = \dfrac{p_U}{a_U - a_0}$, *and* $x_{UD}^h = \dfrac{v}{a_U - a_0}$.

(ii) For each "optimistic" agent $h \in (a_U, 1]$,

(a) the quantities of mortgage contracts lent and borrowed are: $\theta_U^h = 0$ *and* $\varphi_U^h = \frac{1}{1-a_U}$.

(b) the quantities of consumption good are: $x_U^h = 0$, $x_{UU}^h = \frac{u - p_U}{1-a_U}$, *and* $x_{UD}^h = 0$.

3. In state s=D

(i) For each "pessimistic" agent $h \in [0, a_D)$,

(a) the quantities of mortgage contracts lent and borrowed are: $\theta_D^h = \frac{1}{a_D}$ *and* $\varphi_D^h = 0$.

(b) the quantities of consumption good are: $x_D^h = \frac{e}{a_D}$, $x_{UD}^h = \frac{p_D}{a_D}$, and $x_{DD}^h = \frac{d}{a_D}$.

(ii) For each "optimistic" agent $h \in (a_D, a_0)$,

(a) the quantities of mortgage contracts lent and borrowed are: $\theta_D^h = 0$ *and* $\varphi_D^h = \frac{1}{a_0 - a_D}$.

(b) the quantities of consumption good are: $x_D^h = 0$, $x_{UU}^h = \frac{v - p_D}{a_0 - a_D}$, *and* $x_{DD}^h = 0$.

Theorem 2 extends Theorem 1 into a dynamic equilibrium in which the marginal investor beliefs (a_s), the house prices (p_s), and the mortgage prices (q_s) in the three initial states $s \in \{0, U, D\}$ are all endogenously determined. As mentioned above, once the two asset prices (p_s, q_s) are determined, all other endogenous variables of interest, including collateral rate, margins, leverage, and risky interest rate can be pinned down in the model. Consequently, we are able to examine the resulting leverage cycle dynamics as well as the interest rate dynamics.

In addition, Theorem 2 extends the collateral equilibrium of Geanakoplos (2003, 2010a) under risk-free mortgage to a general one under risky mortgage. As such, we can evaluate the dynamics of the aggregate mortgage credit in the leverage cycle. To see this, recall that the total supply of mortgages is fixed at unity (one) in our model, the mortgage price q_s is therefore also the aggregate mortgage debt in state s for the housing market. As such, by examining the evolution of the mortgage price q_s in the three initial states, $s = 0, U, D$, we are able to determine the dynamics of the aggregate mortgage debt as shown in Corollary 1 below.

Corollary 1 (Pro-cyclical Mortgage Debt)

The aggregate mortgage debt in the market is pro-cyclical, i.e., $q_D < q_0 < q_U$.

Corollary 1 indicates that the aggregate mortgage debt increases as the housing market moves up and decreases as it moves down in the leverage cycle. By contrast, such dynamics is void in the model of Geanakoplos (2003, 2010a). Corollary 1 is consistent with the empirical evidence in Schularick and Taylor (2012) that credit growth is pro-cyclical and is a powerful predictor of financial crises over a century, especially post-1945. Thus, the pro-cyclical mortgage credit that emerges from our model is consistent the empirical evidence in the long run.

It is worth noting that although both the house price (p_s) and the mortgage price (q_s) are pro-cyclical in our model, the resulting leverage cycle can be either pro-cyclical or counter-cyclical. To see this, recall that the collateral rate $c_s \equiv \frac{p_s}{q_s}$ as an alternative measure of leverage is determined by the ratio of the two asset prices (p_s, q_s). As the two asset prices move up or down together in the cycle, it is not obvious whether the resulting ratio (or collateral rate) will increase or decrease. Thus, our model is not biased, a priori, to the finding of a pro-cyclical leverage cycle. This shall make any finding of a pro-cyclical leverage cycle in our model significant and meaningful. With this in mind, we proceed to examine whether our model implies pro-cyclical leverage cycle as well as counter-cyclical volatility and interest rate, among others, as evident in the recent financial crisis.

IV. An Investigation of the Special Case of the Model under Risk-free Mortgage

Fostel and Geanakoplos (2012b) show that agents have an incentive to invest in bad volatility projects because such projects offer higher initial price and initial leverage in normal times. As a result, bad news tends to increases volatility and decrease leverage. In this section, we re-examine this issue based on a special case of our model which captures the salient feature of the model in Fostel and Geanakoplos (2012b). Following their setup, we impose two restricted assumptions—risk-free mortgage and extreme payoff structure—to our model. Specifically, our model under risky mortgage can be reduced to a variant of their model under risk-free mortgage simply by replacing the three risky mortgage prices, q_0, q_U, q_D, in Eqs (11)-(13) with the following three risk-free counterparts, while keeping everything else unchanged in our model:

$$q_0 = p_D / (1 + r_f) \tag{19}$$

$$q_U = v / (1 + r_f) \tag{20}$$

$$q_D = d / (1 + r_f) \tag{21}$$

In this special case, the principal of the risk-free mortgage is simply the "low" value of the collateral next period, i.e., p_{sD} for $s = 0, U, D$. Furthermore, following Fostel and Geanakoplos (2012b), we also set the terminal payoff of the "extreme bad volatility" (EBV) project at $(u, v, d) = (1, 1, 0.2)$ and the "extreme good volatility" (EGV) project at $(u, v, d) = (1, 0.2, 0.2)$. Note that in this extreme payoff structure, the EBV's payoffs are volatile only in bad times whereas the EGV's payoffs are volatile only in good times.

The unique equilibrium for this special case of our model is straightforward to obtain by Theorem 2. To illustrate the properties of the two projects (EBV the EGV), we fix the other two exogenous variables at $(e, r_f) = (0.2, 0.05)$ and compute the numerical results of various endogenous variables of interest.[9] We report these numerical results in Table 1, Case 1 (the EBV project) and Case 2 (the EGV project), respectively. In addition, we also report changes in the endogenous variables as the market moves from t=0 to t=1 in Table 2, Cases 1 and 2 for the two projects, respectively.

The numerical results of Cases 1 and 2 in Tables 1 and 2 show that the special case of our model under risk-free mortgage effectively replicates the same dynamics of all variables of interest in Fostel and Geanakoplos (2012b), including (1) extreme optimism, (2) pro-cyclical asset prices, (3) flat interest rates, (4) pro-cyclical leverage of the EBV project and counter-cyclical leverage of the EGV project, and (5) counter-cyclical volatility of the EBV project and pro-cyclical volatility of the EGV project. This suggests that the model of Fostel and Geanakoplos (2012b) is essentially a special case of our model under risk-free mortgage. We discuss the detailed results below.

A. Extreme Optimism

---

[9] We also solve the respective equilibrium based on different sets of exogenous parameters and find our main conclusions do not alter qualitatively.

Comparing Case 1 (the EBV project) to Case 2 (the EGV project), we find that the initial marginal belief (a_0) of the EBV project (0.69) is higher than that of the EGV project (0.57). Second, the marginal belief in state s=U of the EBV project is one at the extreme $(a_U = 1)$. This means that if the market moves up to s=U, only the most optimistic agents $(h = a_U = 1)$ with measure zero will keep the house, while all other agents want to sell their houses. Third, the marginal belief of the EGV project in state s=D $(a_D = 0.57)$ is the same as the initial marginal belief in state s=0 $(a_0 = 0.57)$. Recall that in state s=D, the initial optimists $h \in (a_0, 1]$ default on their payment of mortgage and leave all their housing investment to the initial pessimists $h \in [0, a_0)$. Hence, the marginal belief of the most optimistic agents in s=D is $h = a_0 = a_D = 0.57$, which is at the extreme among the existing agents $h \in [0, a_0)$. This means that if the market moves down to s=D, only the most optimistic agents with measure zero will keep the house, while all other agents want to sell their houses. In either state s=U or s=D, the most optimistic agents with infinite leverage own all housing investment at t=1. Such extreme optimism is due to the assumption of the extreme payoff structure.

B. Pro-cyclical Asset Prices

Cases 1 and 2 show that both projects display pro-cyclical asset prices. Consistent with Fostel and Geanakoplos (2012b), the EBV project offers a higher initial house price (0.77) than the EGV project (0.47). In addition, the EBV project also offers a higher initial mortgage price (0.46) than the EGV project (0.18). These results are driven partly by the higher initial marginal belief of the EBV project and partly by the extreme payoff structure. As will be shown below, these results do not hold in general once we relax the assumption of the extreme payoff structure.

C. Flat Interest Rates

Since we impose risk-free mortgage to accommodate the setup in Fostel and Geanakoplos (2012b), the mortgage interest rate is fixed at the risk-free rate of 5% for both projects regardless of whether the market moves up or down at t=1. Clearly, the interest rate dynamics is void given the assumption of risk-free mortgage.

D. Pro-cyclical Leverage of the EBV Project and Counter-cyclical Leverage of the EGV project

In our model, agents finances their housing investment of p_s with the mortgage of q_s. Thus, the leverage ratio (l_s) and margin requirement (m_s) in the housing investment are endogenously determined as follows:

$$l_s = \frac{p_s}{p_s - q_s};\ m_s = \frac{p_s - q_s}{p_s} = \frac{1}{l_s},\ \text{for } s = 0, U, D \qquad (22)$$

Note that leverage (l_s) and margin (m_s) are simply reciprocal measures to each other as shown in Eq (22) above. Without loss of generality, we thereby focus our discussion on leverage in what follows in order to avoid redundancy.

Comparing Case 1 to Case 2, the EBV project displays pro-cyclical leverage, whereas the EGV project displays counter-cyclical leverage. Specifically, the leverage of the EBV project changes from 2.53 at t=0 to ∞ in state s=U or to 1.64 in state s=D at t=1. The leverage of the EGV project changes from 1.63 at t=0 to 1.36 in state s=U or to ∞ in state s=D at t=1. This result confirms one key finding in Fostel and Geanakoplos (2012b) that agents prefer the EBV project because they can leverage more initially at time t=0.

Note, however, that the infinite leverage in each case is due to the assumption of the extreme payoff structure. One implication of the infinite leverage is that the most optimistic agents in the market in each state s=U or s=D, though with measure zero, can afford to buy the entire housing market as noted above. In the next section, this extreme outcome no longer holds once we relax the assumption of the extreme payoff structure.

E. Counter-cyclical Volatility of the EBV Project and Pro-cyclical Volatility of the EGV project

Given the house price (p_s) and the marginal investor belief (a_s), we can calculate the volatility of the house price in state *s*, denoted by σ_s, as follows,

$$\sigma_s = (p_{sU} - p_{sD})\sqrt{a_s(1-a_s)} \text{ for } s = 0, U, D \qquad (23)$$

Comparing Case 1 to Case 2, the EBV project displays counter-cyclical volatility, whereas the EGV project displays pro-cyclical volatility. Specifically, the volatility of the EBV project changes from 0.22 at t=0 to 0 in state s=U or to 0.39 in state s=D at t=1. The volatility of the EGV project changes from 0.26 at t=0 to 0.37 in state s=U or to 0 in state s=D at t=1. This result confirms another key finding in Fostel and Geanakoplos (2012b) that because agents prefer to invest in the EBV project, bad news leads to higher volatility.

Note also that the zero volatility in each state is again due to the assumption of the extreme payoff structure. That is, the EBV project assumes no volatility in state s=U, whereas the EGV project assumes no volatility in state s=D.

V. The General Properties of the Model under Risky Mortgage

In this section, we relax the two restricted assumptions used in the previous section in order to fully explore the properties of our model under risky mortgage. To this end, we begin by defining a general payoff structure. In our model, all projects are distinguished by their terminal payoffs $(p_{UU,} p_{UD,} p_{DD}) = (u, v, d)$, which reflect the underlying fundamental at t=2. We define a "normal" project as one that has a balanced payoff structure such that $(u, v, d) = (v + \tau, v, v - \tau)$, where $v > 0$ and $\tau > 0$. In contrast, BV and GV projects have unbalanced payoff structure such that the former has a much lower downside payoff while the latter has a much higher upside payoff. To capture such asymmetric patterns, let the payoff of the BV project be $(u, v, d) = (v + \tau, v, v - 3\tau)$ and the payoff of the GV project be $(u, v, d) = (v + 3\tau, v, v - \tau)$. In this setup, the BV project displays higher volatility in state s=D while the GV project displays higher volatility in state s=U, thus capturing the distinction between the two projects in Fostel and Geanakoplos (2012b) without assuming extreme payoffs.

It is worth noting that to an "unbiased" naïve agent who thinks the market will move up or down with equal probability each period, the two projects offer the same expected return at t=0 and the same payoff volatility of 1.5τ at t=2. Thus, the naïve agent will be indifferent in his/her choice between the two projects. This means that there is no bias, a priori, toward selecting either project in our general model. As such, it is meaningful and significant if agents prefer one project over the other in our model.

To illustrate the properties of the two projects in our general model, we fix the payoff parameters at $(v, \tau) = (1, 0.2)$. The resulting terminal payoffs of the BV project at t=2 are $(u, v, d) = (1.2, 1, 0.4)$ and the corresponding payoffs of the GV project are $(u, v, d) = (1.6, 1, 0.8)$. To compute the numerical results of our general model, we fix the other two exogenous parameters at $(e, r_f) = (0.2, 0.05)$ as used in the last section.[10] We report the numerical results of the endogenous variables in Table 1, Case 3 (the BV project) and Case 4 (the GV project), respectively. In addition, we also report changes in the endogenous variables as the market moves from t=0 to t=1 in Table 2, Cases 3 and 4 for the two projects, respectively.

The numerical results show that our model is able to generate the dynamic patterns that are consistent with evidence in the recent financial crisis. Specifically, our model generates (1) pro-cyclical optimism, (2) pro-cyclical asset prices, (3) counter-cyclical interest rates of the BV project, (4) pro-cyclical leverage of the BV project, and (5) counter-cyclical volatility of the BV project. As such, our general model under risky mortgage is able to provide a unified explanation to the phenomenon that bad news increases volatility and interest rate, and decrease optimism, asset prices and leverage. We discuss the detailed results below.

A. Pro-cyclical Optimism

Cases 3 and 4 of Table 1 show that both projects display pro-cyclical optimism. That is, the marginal investor belief (a_s) rises in state s=U or falls in state s=D. In either case, the extreme optimism discovered in the special case above no longer holds. Note also that our result here differs from Fostel and Geanakoplos (2012b) in that the optimism of the EGV project is flat in their model.

B. Pro-cyclical Asset Prices

Cases 3 and 4 of Table 1 show that both projects display pro-cyclical asset prices. Nonetheless, the asset prices of both projects here are much higher than those in the special case above. The higher asset prices are driven mainly by the fact that agents can now borrow more under risky mortgage than under risk-free mortgage. In a sharp contrast to the special case above, the asset prices of the BV project are all less than the prices of the GV projects. This result suggests that a higher initial price of the EBV project in Fostel and Geanakoplos (2012b) is not a general result, but rather is a special case driven by the extreme payoff structure.

C. Counter-cyclical Interest Rates of the BV Project

[10] We also solve the respective equilibrium based on different sets of exogenous parameters and find our main conclusions do not alter qualitatively.

In our general model, agents finance their housing investment with the mortgage of q_s and promise to repay the principal of p_s next period. The resulting mortgage interest rate is:

$$r_s = \frac{p_s}{q_s} - 1, \text{ for } s = 0, U, D \tag{24}$$

Comparing Cases 3 to 4, only the BV project displays counter-cyclical interest rate and thereby appropriately captures the credit condition of the market. Specifically, the interest rate of the BV project changes from 5.94 percent to 5.13 percent in state s=U or to 13.63 percent in state s=D. In contrast, the interest rate of the GV project always moves down from t=0 to t=1 regardless of the market condition at t=1.

D. Pro-cyclical Leverage of the BV Project

Comparing Cases 3 to 4, only the BV project displays pro-cyclical leverage. Specifically, the leverage of the BV project changes from 17.83 to 20.50 in state s=U or to 8.34 in state s=D. In contrast, the leverage of the GV project always moves up from t=0 to t=1 regardless of the market condition at t=1. This result suggests that the pro-cyclical leverage of the EBV project discovered in Fostel and Geanakoplos (2012b) is robust in our model.

Moreover, the initial leverage of the BV project (17.83) is greater than that of the GV project (15.06) in our model. This result confirms again a key finding in Fostel and Geanakoplos (2012b) that agents prefer the EBV project because they can leverage more initially at time t=0. Note also that the extreme outcome of the infinite leverage in the special case no longer holds here once we relax the extreme payoff structure in our general model.

E. Counter-cyclical Volatility of the BV Project

Comparing Cases 3 to 4, only the BV project displays counter-cyclical volatility. Specifically, the volatility of the BV project changes from 0.06 at t=0 to 0.02 in state s=U or to 0.21 in state s=D at t=1. In contrast, the leverage of the GV project always moves down from t=0 to t=1 regardless of the market condition at t=1. This result confirms again another key finding in Fostel and Geanakoplos (2012b) that because agents prefer to invest in the BV project, bad news leads to higher volatility.

Table 1 also shows that the initial volatility is much lower in our model under risky mortgage than that in the special case under risk-free mortgage. For example, the initial volatility of the BV project is 0.22 under risk-free mortgage in Case 1 and it is 0.06 under risky mortgage in Case 3. This result suggests that risk-free mortgage actually leads to more volatile housing market. Indeed, Table 2 shows that the house price of the BV project drops 36 percent from 0.77 to 0.49 in Case 1 under risk-free mortgage and it only drops 19 percent from 1.07 to 0.87 in Case 3 under risky mortgage. Note also that the extreme outcome of zero volatility in the special case no longer holds here once we relax the extreme payoff structure in the special case above.

VI. The Extended Model with Double Leverage Cycle

Geanakoplos (2010a, 2010b) posits that the recent financial crisis is particularly bad because it suffers from a "double leverage cycle" problem. He argues that the current leverage cycle is really a double leverage cycle: one in the housing market via mortgages and the other in the mortgage securities market via the repo market, and the two reinforce each other in a positive feedback loop. In this double leverage cycle, the same collateral (house) backs the mortgage payment first and then backs the mortgage securities again. On the one hand, the crash of the housing market not only has a direct impact on mortgage payments, but also ripples through the mortgage securities market. On the other hand, the crash of the mortgage securities market adversely affects the loan the homeowner can get, which in turn causes problems to the housing market. In this section, we incorporate such a double leverage cycle into our model and examine whether this does lead to more severe leverage cycle and greater volatility in the housing market and hence a worse financial crisis.

In our current setup, there is a primary leverage cycle in the housing market in which the leverage is given by $l_s = \dfrac{p_s}{p_s - q_s}$, for $s = 0, U, D$. That is, the leverage of the housing investment evolves through the cycle as the market condition changes over time. In particular, the leverage rises at good times and falls at bad times, i.e., $l_D < l_0 < l_U$.

We introduce now a secondary leverage cycle in the mortgage securities market. In this market, the collateral of the mortgage is used again as the collateral in the repo market to obtain financing, which in turn is used as the source of fund for issuing the mortgage in the first place. Following the standard practice in the repo market, such financing is done with some haircut (or margin) of the collateral. In other words, the borrower in the repo market cannot borrow the full amount of the collateral (house) value, but rather at some discount. As such, the borrower of fund in the repo market, who is also the lender of the mortgage in the housing market, can only lend at most what he or she can borrow from the repo market. In this spirit, let the amount of the fund borrowed in the repo market (which is also the principal or face value of the mortgage in the housing market), denoted by F_s for state $s = 0, U, D$, be a weighted average of the current collateral (house) value, p_s, and the recovery value of the collateral in the down market next period, p_{sD}, i.e., the mortgage principal is given by

$$F_s = (1-n)p_s + n \cdot p_{sD}, \text{ where } s = 0, U, D \text{ and } 0 \le n \le 1 \tag{25}$$

We add this secondary leverage cycle into our model simply by replacing the mortgage principal p_s in (11)-(13) with the updated principal F_s for state $s = 0, U, D$ in Eq (25). This results in three contingent mortgage prices under the double leverage cycle as follows:

$$q_0 = (a_0 \cdot ((1-n)p_0 + n \cdot p_D) + (1-a_0)p_D) / (1+r_f) \tag{26}$$

$$q_U = (a_U \cdot ((1-n)p_U + n \cdot v) + (1-a_U)v) / (1+r_f) \tag{27}$$

$$q_D = (a_D \cdot ((1-n)p_D + n \cdot d) + (1-a_D)d) / (1+r_f) \tag{28}$$

Note that the range of the mortgage principal F_s defined in (25) is between p_{sD} and p_s given that $p_{sD} < p_s$. This definition of the principal includes two special cases considered in the previous sections: (1) for $n = 0$, it corresponds to our model under risky mortgage with a principal equal to the value of the collateral p_s in state *s* and (2) for $n = 1$, it corresponds to Geanakoplos (2003, 2010a) and Foster and Geanakoplos (2012b) under risk-free mortgage with a principal equal to the recovery value of the collateral p_{sD} in state *sD*.

In this setup, the margin of the secondary cycle is $(p_s - F_s) / p_s = n(p_s - p_{sD}) / p_s$. To the extent that the recovery value of the collateral, p_{sD}, is negligible relative to the current value of the collateral, p_s, the weight parameter *n* may serve as a crude measure of the margin of the secondary cycle. In this sense, we call parameter *n* the "funding margin" in the secondary leverage cycle. This funding margin is useful to gauge the credit condition or "funding liquidity" in the mortgage securities market. In particular, an increase in the funding margin *n* corresponds to a tightening of the credit or funding liquidity in the mortgage securities market.

Brunnermeier and Pedersen (2009) distinguish liquidity risks between market liquidity and funding liquidity. Market liquidity risk captures the inability of an asset to be sold without causing a significant movement in its price and with minimum loss of value, whereas funding liquidity risk captures the inability of a financial intermediary to service its liabilities as they fall due or can only be met at an uneconomic price. In our double leverage cycle, we are able to distinguish between these two kinds of liquidity risks. On the one hand, the "funding margin" (n) in our model captures the funding liquidity risk as noted above.[11] On the other hand, the margin requirement (m_s) defined in Eq (22) effectively captures the market liquidity risk since it is jointly determined by the two asset prices (p_s, q_s). An increase (decrease) in the margin requirement means that the value of the mortgage falls (rises) more proportionally than that of its collateral as the market moves down (up) due to a heightened (lessened) credit condition. In this sense, we call margin requirement (m_s) the "market margin" and use it to gauge market liquidity risk. In effect, this market margin (m_s) gauges the tightness of the primary leverage cycle in the housing market, whereas the funding margin (n) gauges the tightness of the secondary leverage cycle in the mortgage securities market.

Since the mortgage principal $F_0 = (1-n)p_0 + n \cdot p_D$ will only be paid off in state *s*=U, this leads to a corresponding change of the market clearing condition in state *s*=U such that the clearing condition in Eq (15) is replaced by its counterpart under double leverage cycle below:

$$\frac{1-a_U}{1-a_0}(e-((1-n)p_0 + n \cdot p_D)) + q_U = \frac{a_U - a_0}{1-a_0} p_U \tag{29}$$

All other equations in our model, i.e., the three contingent house prices in Eqs (8)-(10) and the two other market clearing conditions in Eqs (17)-(18), remain the same. As such, given a fixed funding margin *n* such that $0 \le n \le 1$, a unique equilibrium of the extended model with the double leverage cycle can be obtained as an expanded version of Theorem 2.

To illustrate the properties of the equilibrium of the extended model, we fix the "funding margin" at $n = 0.2$ and compute the numerical results based on the same exogenous parameters at $(e, r_f) = (0.2, 0.05)$ and the same payoffs of the BV project at $(u, v, d) = (1.2, 1, 0.4)$ and of the

[11] Here the funding margin *n* is deterministic. Our model can be expanded further so that the funding margin can change dynamically as the two liquidity risks reinforce each other in a feedback loop. To conserve space, we leave it out of the paper for future research.

GV project at $(u,v,d)=(1.6,1,0.8)$.[12] We report the numerical results of the endogenous variables under double leverage cycle in Table 1, Case 5 (the BV project) and Case 6 (the GV project), respectively. In addition, we also report changes in the endogenous variables as the market moves from t=0 to t=1 in Table 2, Cases 5 and 6 for the two projects, respectively.

The emphasis of our analysis in this section is twofold. First, by comparing Case 5 (the BV project) to Case 6 (the GV project), we examine whether our findings under single leverage cycle above are robust to the setting of double leverage cycle. Second, by comparing Case 3 (funding margin *n*=0) to Case 5 (funding margin *n*=0.2), both of the BV project, we study what is the marginal effect of tightening the funding margin *n* in the secondary leverage cycle on the primary leverage cycle.

The numerical results under double leverage cycle below clearly confirm the robustness of our findings under single leverage cycle above. In particular, agents still prefer the BV project over the GV project in this extended model under double leverage cycle. Moreover, the BV project still generates pro-cyclical optimism, asset prices and leverage, and counter-cyclical volatility and interest rate. Thus, our extended model strengthens the findings in Fostel and Geanakoplos (2012b) and provides a robust and unified explanation to the phenomenon that bad news increases volatility and interest rate, and decrease optimism, asset prices and leverage.

In addition, the numerical results show that tightening the funding margin in the secondary leverage cycle dampens optimism, asset prices and leverage, whereas it raises volatility and interest rate. Furthermore, such tightening policy magnifies the leverage cycle as manifested in more volatile dynamics of the endogenous variables as the market moves from t=0 to t=1. Hence, double leverage cycle leads to more volatile markets and severe leverage cycle, thus resulting in worse financial crises as posited in Geanakoplos (2010a, 2010b). We discuss the marginal effect of the double leverage cycle in more details below.

A. Lower but More Volatile Optimism

Cases 5 and 6 show that both projects continue to display pro-cyclical optimism. We find that while tightening the funding margin lowers the level of optimism, it magnifies its changes.

[12] We have also run numerical analyses with different values of *n*'s for the double leverage cycle and the results remain the same qualitatively.

For example, when the market moves down from t=0 to t=1, the marginal belief (a_s) drops 9 percent from 0.95 to 0.86 in Case 3 and it drops 20 percent from 0.91 to 0.73 in Case 5.

B. Lower but More Volatile Asset Prices of the BV Project

Cases 5 and 6 show that both projects continue to display pro-cyclical asset prices. We also find that while tightening the funding margin lowers the level of asset prices for both projects, it magnifies the changes of the BV project. For example, when the market moves down from t=0 to t=1, the house price (p_s) drops 19 percent from 1.07 to 0.87 in Case 3 and it drops 24 percent from 1.05 to 0.80 in Case 5.

In addition, the asset prices of the BV project are still less than the prices of the GV projects. This result confirms again that a higher initial price of the EBV project in Fostel and Geanakoplos (2012b) is not a general result, but rather is a special case driven by the extreme payoff structure.

Note further that asset prices in Cases 5 and 6 are still higher than the prices in Cases 1 or 2 under risk-free mortgage. This confirms again the positive relation between asset prices and leverage because agents can borrow more under risky mortgage. But, tightening the funding margin from zero percent under single leverage cycle in Cases 3 and 4 to 20 percent under double leverage cycle in Cases 5 and 6 will curtail such borrowing capacity and thereby result in lower asset prices in latter cases.

C. Higher and More Volatile Interest Rate of the BV Project

Comparing Case 5 to Case 6, only the BV project displays counter-cyclical interest rate. For the BV project, we also find that tightening the funding margin raises both the level and changes of interest rate. For example, when the market moves down from t=0 to t=1, the interest rate (r_s) jumps 129 percent from 0.0594 to 0.1363 in Case 3 and it jumps 174 percent from 0.0705 to 0.1935 in Case 5. This result highlights the positive relation between funding margin in the secondary cycle and mortgage interest rate in the primary cycle.

D. Lower but More Volatile Leverage of the BV Project

Comparing Cases 4 to 5, only the BV project displays pro-cyclical leverage. Moreover, the initial leverage of the BV project (8.98) is greater than that of the GV project (7.18) in our extended model. These results are consistent with Fostel and Geanakoplos (2012b) and lead to the same conclusion that agents prefer the BV project because they can leverage more initially at time t=0.

For the BV project, we also find that while tightening the funding margin lowers the level of leverage, it magnifies its changes. For example, when the market moves up from t=0 to t=1, leverage (l_s) jumps 15 percent from 17.83 to 20.50 in Case 3 and it jumps 53 percent from 8.98 to 13.71 in Case 5. This result confirms that an increase in funding margin in the secondary cycle does lead to higher market margin and hence lower leverage in the primary cycle; at the same time, it also leads to more severe leverage cycle as posited in Geanakoplos (2010a, 2010b).

E. Higher Level of Volatility

Comparing Cases 5 to 6, only the BV project displays counter-cyclical volatility. This result confirms again another key finding in Fostel and Geanakoplos (2012b) that because agents prefer to invest in the BV project, bad news leads to higher volatility. For both projects, we also find that tightening the funding margin raises the level of volatility. For example, the initial volatility increases from 0.06 in Case 3 to 0.10 in Case 5 and it increases from 0.14 in Case 4 to 0.19 in Case 6. This result highlights the positive relation between funding margin in the secondary cycle and volatility in the primary cycle.

VII. Conclusion

The recent financial crisis is rooted in a joint collapse of the mortgage and the housing markets. This crisis highlights that mortgage is a risky derivative of its underlying housing collateral since both assets are subject to the same underlying fundamental and default risk. In this setting, we combine the no-arbitrage valuation approach with the equilibrium valuation approach to develop a dynamic model in which both leverage cycle and interest rate are endogenously determined in equilibrium. The model thus extends that of Geanakoplos (2003, 2010a) under risk-free mortgage to one under risky mortgage.

Following Fostel and Geanakoplos (2012b), we investigate a special case of our model under the assumptions of risk-free mortgage and extreme payoff structure. In this special case, we confirm their conclusion that agents prefer the extreme BV project because it offers higher initial price and leverage in normal times. However, once we relax the two restricted assumptions in our general model, we find that the BV project only offers higher initial leverage, but not higher initial price. Nonetheless, to the extent that agents prefer higher initial leverage, they will still choose to invest in the BV project, which generates pro-cyclical optimism, asset prices and leverage, and counter-cyclical volatility and interest rate. Thus, our general model provides a robust and unified explanation why bad news increases volatility and interest rate, and decreases optimism, asset prices and leverage.

Finally, we add a secondary leverage cycle in the mortgage securities market to our model of the primary leverage cycle in the housing market. The extended model thus captures the "double leverage cycle" noted in Geanakoplos (2010a, 2010b). We find that all our findings under single leverage cycle remain intact in the extended model under double leverage cycle. In addition, we find that tightening the funding margin in the secondary leverage cycle dampens optimism, asset prices and leverage, whereas it raises volatility and interest rate. Moreover, such tightening policy magnifies leverage cycle as the market condition changes. Hence, double leverage cycle leads to more volatile markets and severe leverage cycle, resulting in worse financial crises. To investigate these new testable implications of our extended model is beyond the current scope of our paper, but it certainly can induce useful future research concerning the impact of double leverage cycle on systemic risk and financial stability in the economy.

Appendix: Proofs

The Proof of Theorem 1

Given the linear utility and the heterogeneous beliefs such that each agent $h \in H = [0,1]$ is identified by his/her subjective probability *h* for the good state s=U, we conjecture that there exists some marginal agent $a \in (0,1)$ who will be indifferent between buying and selling Y at t=0. Furthermore, given the constants *e*, r_f, *u and d*, the marginal agent's belief *a*, the house price *p*, and the mortgage price *q* are uniquely determined as shown in (7).

Since only the optimistic agents (i.e., $h > a$) will borrow the mortgage Z and only the pessimistic agents (i.e., $h < a$) will lend the mortgage Z, the mortgage market is cleared as follows,

$$\sum_{h \in H} \theta^h = \sum_{h<a} \theta^h = a \cdot \frac{1}{a} = (1-a)\frac{1}{1-a} = \sum_{h>a} \varphi^h = \sum_{h \in H} \varphi^h \qquad (30)$$

At time 0, the pessimistic agents (i.e., $h < a$) consume all endowments, while the optimistic agents (i.e., $h > a$) consume none. As a result, the commodity X market at time 0 is cleared as follows,

$$\sum_{h \in H} x_0^h = \sum_{h<a} x_0^h + \sum_{h>a} x_0^h = a \cdot \frac{1}{a} e + (1-a)0 = 1 \cdot e = \sum_{h \in H} e \qquad (31)$$

At time 1, there are two possible states. In state U, the pessimistic agents (i.e., $h < a$) get the full payment of the mortgage loan, whereas the optimistic agents (i.e., $h > a$) keep the residue payoff of the house after paying the mortgage Z in full, i.e., *u*-1 of commodity X per mortgage contract borrowed. As a result, the commodity X market in state U at time 1 is cleared as follows,

$$\sum_{h \in H} x_U^h = \sum_{h<a} x_U^h + \sum_{h>a} x_U^h = a \cdot \frac{1}{a} + (1-a)\frac{1}{1-a}(u-1) = u = (1-a)\frac{1}{1-a}u = \sum_{h>a} \varphi^h u \qquad (32)$$

In state D, the optimistic agents (i.e., $h > a$) default and hence get nothing from the house Y, while the pessimistic agents (i.e., $h < a$) get all payoff from the house Y, thus partially recovering their mortgage Z with *d* unit of commodity X per mortgage contract lent. As a result, the commodity X market in state D at time 1 is cleared as follows,

$$\sum_{h\in H} x_D^h = \sum_{h<a} x_D^h + \sum_{h>a} x_D^h = a\cdot\frac{1}{a}d + (1-a)0 = d = (1-a)\frac{1}{1-a}d = \sum_{h>a}\varphi^h d \qquad (33)$$

Finally, the expected utility of agent *a* if he sells, denoted by U_-^a, is given by

$$U_-^a = x_0^h + \frac{1}{1+r_f}(a\cdot x_U^h + (1-a)\cdot x_D^h) = \frac{1}{a}e + \frac{1}{1+r_f}(a\cdot\frac{1}{a} + (1-a)\frac{1}{a}d) = \frac{1}{a}e + \frac{1}{1+r_f}(1+\frac{1-a}{a}d) \; (34)$$

The expected utility of agent *a* if he buys, denoted by U_+^a, is given by

$$U_+^a = x_0^h + \frac{1}{1+r_f}(a\cdot x_U^h + (1-a)\cdot x_D^h) = 0 + \frac{1}{1+r_f}(a\cdot\frac{1}{1-a}(u-1)+0) = \frac{1}{1+r_f}(\frac{a}{1-a}(u-1)) \; (35)$$

The marginal agent *a* is indifferent between buying and selling Y if the two utilities are equal. To see this, compute the difference between the two utilities as follows:

$$U_-^a - U_+^a = \frac{1}{a}(e + \frac{1}{1+r_f}(a+(1-a)d) - \frac{1}{1+r_f}\frac{a^2}{1-a}(u-1)) = \frac{1}{a(1-a)}((1-a)e + q - a\cdot p) = 0 \; (36)$$

The last equality is derived from the market clearing condition of the housing market as given in (6). Thus, the conjecture that the marginal agent is indifferent between buying and selling Y at t=0 is consistent with the equilibrium that emerges. Q.E.D.

<u>The Proof of Theorem 2</u>

1. In state s=D

Given $a_0 \in (0,1)$ from the last period, substituting p_D and q_D from (10) and (13) into (17) and rearranging terms, one obtains a quadratic function $G(a_D)$ in a_D such that

$$G(a_D) = (1-\frac{a_0}{1+r_f})(v-d)(a_D)^2 + ((1+r_f)e + d(1+(1-\frac{1}{1+r_f})a_0))a_D - a_0((1+r_f)e+d) = 0 \;\; (37)$$

Given $e>0,\ r_f>0,\ 0<d<v$ and $0<a_0<1$, the quadratic equation in (37) has two real roots, one positive and one negative. Furthermore, evaluating $G(a_D)$ at $a_D = 0$ and $a_D = a_0$, respectively, one obtains:

$$G(0) = -a_0((1+r_f)e+d) < 0 \qquad (38)$$

$$G(a_0) = a_0^2((1-\frac{a_0}{1+r_f})(v-d) + (1-\frac{1}{1+r_f})d) > 0 \qquad (39)$$

Since $G(a_D)$ is a continuous function in a_D, there exists a unique solution $a_D^* \in (0, a_0)$ such that $G(a_D^*) = 0$. This solution a_D^* is precisely the unique positive root in (37), given $a_0 \in (0,1)$. This solution a_D^* is thus a function of $a_0 \in (0,1)$.

2. In state s=U

Given $a_0 \in (0,1)$ from the last period, substituting p_0, p_U, p_D and q_U from (8), (9), (10) and (12) into (15) and rearranging terms, one obtains a quadratic function $H(a_U)$ in a_U such that

$$\begin{aligned} H(a_U) &= r_f(u-v)a_U^2 + (r_f(2v - a_0 u) + (1-a_0)(1-a_D)(v-d) + (1+r_f)^2 e)a_U \\ &-(r_f v + (1-a_0)(1-a_D)(v-d) + (1+r_f)^2 e) = 0 \end{aligned} \qquad (40)$$

Given $e > 0$, $r_f > 0$, $0 < d < v < u$ and $0 < a_D < a_0 < 1$, the quadratic equation in (37) has two real roots, one positive and one negative. Furthermore, evaluating $G(a_U)$ at $a_U = a_0$ and $a_U = 1$, respectively, one obtains:

$$H(a_0) = -(1-a_0)^2 r_f v - (1-a_0)((1-a_0)(1-a_D)(v-d) + (1+r_f)^2 e) < 0 \qquad (41)$$

$$H(1) = (1-a_0) r_f u > 0 \qquad (42)$$

Since $H(a_U)$ is a continuous function in a_U, there exists a unique solution $a_U^* \in (a_0, 1)$ such that $H(a_U^*) = 0$. This solution a_U^* is precisely the unique positive root in (40), given $a_0 \in (0,1)$. This solution a_U^* is thus a function of $a_0 \in (0,1)$.

3. In state s=0

Given $e > 0$, $r_f > 0$, $0 < d < u$, and p_0, p_U, p_D, q_0 from (8)-(11), rearranging terms in (18), one obtains a function $F(a_0)$ in a_0 such that

$$F(a_0) = a_0(p_0 + e) - (q_0 + e) = 0 \qquad (43)$$

Now evaluating $\lim_{a_0 \to 0} F(a_0)$ and $\lim_{a_0 \to 1} F(a_0)$ as follows:

As $a_0 \to 0$, we have $a_D^* \to 0$ and hence $p_D \to \frac{d}{1+r_f}$, $p_0 \to \frac{d}{(1+r_f)^2}$, and $q_0 \to \frac{d}{(1+r_f)^2}$. As a result, $\lim_{a_0 \to 0} F(a_0) = -(\frac{d}{(1+r_f)^2} + e) < 0$.

As $a_0 \to 1$, we have $a_U^* \to 1$ and hence $p_U \to \frac{u}{1+r_f}$, $p_0 \to \frac{u}{(1+r_f)^2}$, and $q_0 \to \frac{u}{(1+r_f)^3}$. As a result, $\lim_{a_0 \to 1} F(a_0) = (\frac{u}{(1+r_f)^2} + e) - (\frac{u}{(1+r_f)^3} + e) = \frac{u}{(1+r_f)^2}(1 - \frac{1}{1+r_f}) > 0$.

Since $F(a_0)$ is continuous, there must be an $a_0^* \in (0,1)$ such that $F(a_0^*) = 0$. Note further that the continuous function $F(a_0)$ is increasing in a_0 in a neighborhood of $F(a_0) = 0$. Thus, there is at most a unique $a_0 \in (0,1)$ such that $F(a_0) = 0$. Taken together, there exists a unique solution $a_0^* \in (0,1)$ such that $F(a_0^*) = 0$.

Since both the solution $a_D^* \in (0, a_0)$ and the solution $a_U^* \in (a_0, 1)$ above are functions of $a_0 \in (0,1)$, the uniqueness of the solution $a_0^* \in (0,1)$ also ensures the uniqueness of a_D^* and a_U^* such that $0 < a_D^* < a_0^* < a_U^* < 1$.

4. The mortgage market clearing condition for each state $s = (0, U, D)$ is satisfied as follows:

$$\sum_{h \in H} \theta_0^h = \sum_{h < a_0} \theta_0^h = a_0 \cdot \frac{1}{a_0} = (1 - a_0)\frac{1}{1 - a_0} = \sum_{h > a_0} \varphi_0^h = \sum_{h \in H} \varphi_0^h, \text{ for } s = 0 \qquad (44)$$

$$\sum_{h \in (a_0, 1]} \theta_U^h = \sum_{h \in (a_0, a_U)} \theta_U^h = (a_U - a_0) \cdot \frac{1}{a_U - a_0} = (1 - a_U)\frac{1}{1 - a_U} = \sum_{h \in (a_U, 1]} \varphi_U^h = \sum_{h \in (a_0, 1]} \varphi_U^h, \text{ for } s = U \; (45)$$

$$\sum_{h \in [0, a_0)} \theta_D^h = \sum_{h \in [0, a_D)} \theta_D^h = a_D \cdot \frac{1}{a_D} = (a_0 - a_D)\frac{1}{a_0 - a_D} = \sum_{h \in (a_D, a_0)} \varphi_D^h = \sum_{h \in [0, a_0)} \varphi_D^h, \text{ for } s = D \; (46)$$

Note that (44)-(46) are obtained based on the same method used in (30). Likewise, the commodity market clearing condition for each state $s = (0, U, D, UU, UD, DD)$ is obtained based on the method used in (31)-(33). To conserve space, we leave it to the reader.

5. Lastly, we verify the conjecture that the marginal agents (a_0, a_U, a_D) are indifferent between buying and selling the house in their corresponding states s=(0,U,D) in the equilibrium. Since the proof follows the same approach for all three states, to avoid repetition and conserve space, we show the case for state s=0 and leave the other two states, s=U and s=D, for the readers.

In state s=0, the expected utility of agent a_0 if he sells, denoted by $U_-^{a_0}$, is given by

$$U_{-}^{a_0}=x_0^h+\frac{1}{1+r_f}(a_0\cdot x_U^h+(1-a_0)\cdot x_D^h)=\frac{1}{a_0}(e+\frac{1}{1+r_f}(a_0\cdot p_0+(1-a_0)p_D)=\frac{1}{a_0}(e+q_0)\quad(47)$$

The expected utility of agent a_0 if he buys, denoted by $U_{+}^{a_0}$, is given by

$$U_{+}^{a_0}=x_0^h+\frac{1}{1+r_f}(a_0\cdot x_U^h+(1-a_0)\cdot x_D^h)=\frac{1}{1-a_0}\frac{1}{1+r_f}(a_0(p_U-p_0))=\frac{1}{1-a_0}(p_0-q_0)\quad(48)$$

The marginal agent a_0 is indifferent between buying and selling Y if the two utilities are equal.

$$U_{-}^{a_0}-U_{+}^{a_0}=\frac{1}{a_0}(e+q_0)-\frac{1}{1-a_0}(p_0-q_0)=\frac{1}{a_0(1-a_0)}((1-a_0)e+q_0-a_0\cdot p_0)=0\quad(49)$$

The last equality in (49) is derived from the market clearing condition in (18). Thus, the conjecture that the marginal agent is indifferent between buying and selling the house Y is indeed consistent with the equilibrium that emerges. Q.E.D.

The Proof of Corollary 1

1. Check $q_0>q_D$

$$\begin{aligned}&(1+r_f)(q_0-q_D)\\&=(a_0p_0+(1-a_0)p_D)-(a_Dp_D+(1-a_D)d)=a_0(p_0-p_D)+(1-a_D)(p_D-d)>0\end{aligned}\quad(50)$$

The inequality in (50) is due to $0<a_D<a_0<1$ and $d<p_D<p_0$.

2. Check $q_U>q_0$

$$\begin{aligned}&(1+r_f)(q_U-q_0)\\&=(a_Up_U+(1-a_U)v)-(a_0p_0+(1-a_0)p_D)=a_U(p_U-v)+(v-p_D)-a_0(p_0-p_D)\\&>a_U(p_U-v)+a_U(v-p_D)-a_0(p_0-p_D)=a_U(p_U-p_D)-a_0(p_0-p_D)>0\end{aligned}\quad(51)$$

The last inequality in (51) is due to $0<a_0<a_U<1$ and $p_D<p_0<p_U$. Q.E.D.

References

Acharya, V. and S. Viswanathan (2011), Leverage, moral hazard and liquidity, *Journal of Finance*, 66, 99-138.

Adrian, T. and H. Shin (2010), Liquidity and leverage, *Journal of Financial Intermediation*, 19, 418-437.

Araujo, A., F. Kubler, and S. Schommer (2012), Regulating collateral-requirements when markets are incomplete, *Journal of Economics Theory*, 147, 450-476.

Bernanke, B., M. Gertler, and S. Gilchrist (1999), The financial accelerator in a quantitative business cycle framework, in J. Tayolor and M. Woodford, ed., *Handbook of macroeconomics*, Elsevier, 1, 1341-1393.

Brunnermeier, M. and L. Pedersen (2009), Market liquidity and funding liquidity, *Review of Financial Studies*, 22, 2201-2238.

Caballero, R. and A. Krishnamurthy (2001), International and domestic collateral constraints in a model of emerging market crises, *Journal of Monetary Economics*, 48, 513-548.

Fostel, A. and J. Geanakoplos (2008a), Collateral restrictions and liquidity under-supply: a simple model, *Economic Theory*, 35, 441-467

Fostel, A. and J. Geanakoplos (2008b), Leverage cycles and the anxious economy, *American Economic Review*, 98, 1211-1244.

Fostel, A. and J. Geanakoplos (2012a), Tranching, CDS, and asset prices: how financial innovation can cause bubbles and crashes, *American Economic Journal: Macroeconomics*, 4, 190-225

Fostel, A. and J. Geanakoplos (2012b), Why does bad news increase volatility and decrease leverage? *Journal of Economic Theory*, 147, 501-525.

Garleanu N. and L. Pedersen (2011), Margin-based asset pricing and deviations from the law of one price, *Review of Financial Studies*, 24, 1980-2022.

Geanakoplos, J. (1997), Promises, Promises, in B. Authur, S. Durlauf, and D. Lane, ed., *The economy as an evolving complex system II*, Reading MA: Addison-Wesley, 285-320.

Geanakoplos, J. (2003), Liquidity, default, and crashes, endogenous contracts in general equilibrium, in M. Dewatripont, L. Hansen, and S. Turnovsky, ed., *Advances in Economics and*

Econometrics: Theory and Applications, Eighth World Conference, Econometric Society Monographs, Cambridge, MA: Cambridge University Press, 2, 170-205.

Geanakoplos, J. (2010a), The leverage cycle, in D. Acemoglu, K. Rogoff, and M. Woodford, ed., *NBER Macroeconomic Annual*, Cambridge, MA: MIT Press, 1-65.

Geanakoplos, J. (2010b), Solving present crisis and managing the leverage cycle, in *Federal Reserve Bank of New York Economic Policy Review*, 101-131.

Geanakoplos, J. and W. Zame (1997), Collateralized security markets, CFDP Working Paper.

Gromb D. and D. Vayanos (2002), Equilibrium and welfare in markets with financially constrained arbitrageurs, *Journal of Financial Economics*, 66, 361-407.

He, Z. and W. Xiong (2012), Debt financing in asset markets, *American Economic Review*, 102, 88-94.

Holmstrom and Tirole (1997), Financial Intermediation, loanable funds, and the real sector, *Quarterly Journal of Economics*, 112, 663-691.

Kiyotaki N. and J. Moore (1997), Credit cycles, *Journal of Political Economy*, 105, 211-248.

Schularick, M. and A. Taylor (2012), Credit booms gone bust: monetary policy, leverage cycle, and financial crises, 1870-2008. *American Economic Review*, 102, 1029-1061.

Shleifer A. and R. Vishny (1992), Liquidation values and debt capacity: a market equilibrium approach, *Journal of Finance*, 47, 1343-1366.

Simsek, A. (2013), Belief disagreements and collateral constraints, *Econometrica*, 81, 1-53.

Wang, F. and T. Zhang (2013), Financial crisis and credit crunch in the housing market, *The Journal of Real Estate Finance and Economics*, *forthcoming.*

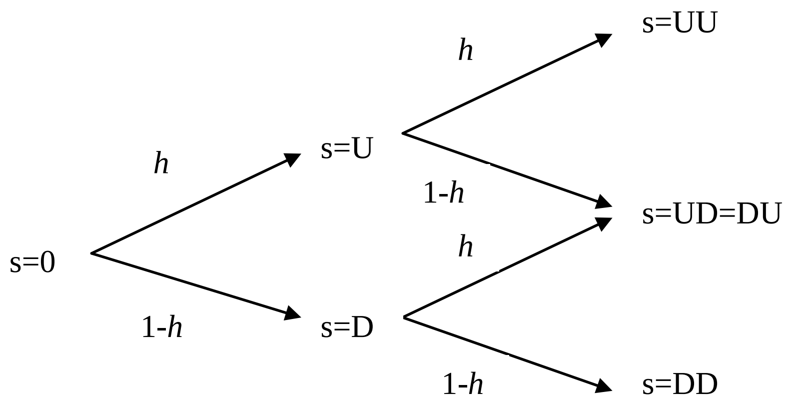

Figure 1: The State of Nature Tree

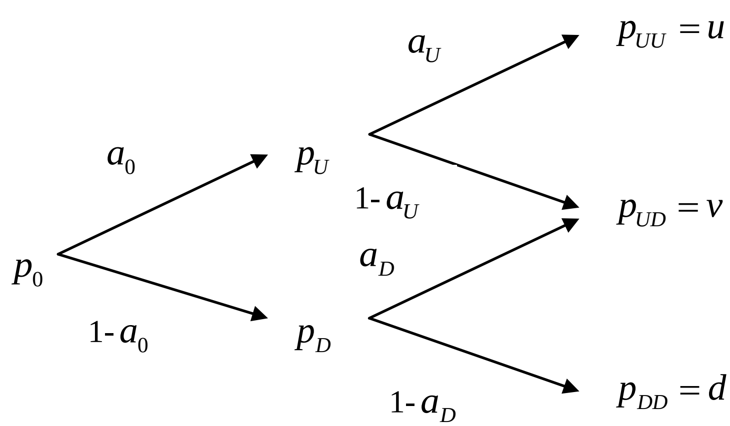

Figure 2: The Contingent House Price Tree

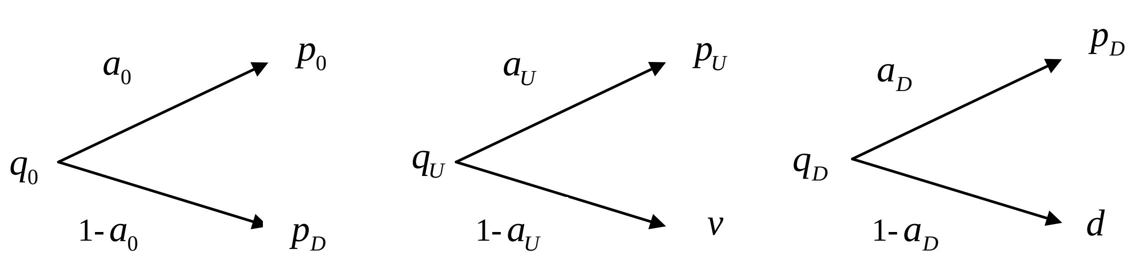

Figure 3: The Contingent Mortgage Price Trees

Table 1: The Leverage Cycle

Variable	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
a_0	0.69	0.57	0.95	0.94	0.91	0.88
a_U	1.00	0.69	0.99	0.98	0.98	0.95
a_D	0.39	0.57	0.86	0.89	0.73	0.81
p_0	0.77	0.47	1.07	1.41	1.05	1.36
p_U	0.95	0.72	1.14	1.52	1.14	1.49
p_D	0.49	0.19	0.87	0.93	0.80	0.92
q_0	0.46	0.18	1.01	1.32	0.94	1.17
q_U	0.95	0.19	1.09	1.44	1.06	1.31
q_D	0.19	0.19	0.77	0.87	0.60	0.83
r_0	0.0500	0.0500	0.0594	0.0711	0.0705	0.0865
r_U	0.0500	0.0500	0.0513	0.0555	0.0524	0.0659
r_D	0.0500	0.0500	0.1363	0.0661	0.1935	0.0709
l_0	2.53	1.63	17.83	15.06	8.98	7.18
l_U	∞	1.36	20.50	19.03	13.71	8.07
l_D	1.64	∞	8.34	16.14	4.07	11.11
σ_0	0.22	0.26	0.06	0.14	0.10	0.19
σ_U	0.00	0.37	0.02	0.07	0.03	0.13
σ_D	0.39	0.00	0.21	0.06	0.27	0.08

Notes:

Case 1: Riskfree debt and EBV project: $(u,v,d,n)=(1,1,0.2,1)$

Case 2: Riskfree debt and EGV project: $(u,v,d,n)=(1,0.2,0.2,1)$

Case 3: Risky debt and BV project: $(u,v,d,n)=(1.2,1,0.4,0)$

Case 4: Risky debt and GV project: $(u,v,d,n)=(1.6,1,0.8,0)$

Case 5: Risky debt, BV project, and double leverage cycle: $(u,v,d,n)=(1.2,1,0.4,0.2)$

Case 6: Risky debt, GV project, and double leverage cycle: $(u,v,d,n)=(1.6,1,0.8,0.2)$

Common Parameters: $(e,r_f)=(0.2,0.05)$

Table 2: The Changes in the Leverage Cycle

Change	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
$a_U / a_0 - 1$	45%	21%	4%	4%	8%	8%
$a_D / a_0 - 1$	-43%	0%	-9%	-5%	-20%	-8%
$p_U / p_0 - 1$	23%	53%	7%	8%	9%	10%
$p_D / p_0 - 1$	-36%	-60%	-19%	-34%	-24%	-32%
$q_U / q_0 - 1$	107%	6%	8%	9%	13%	12%
$q_D / q_0 - 1$	-59%	6%	-24%	-34%	-36%	-29%
$r_U / r_0 - 1$	0%	0%	-14%	-22%	-26%	-24%
$r_D / r_0 - 1$	0%	0%	129%	-7%	174%	-18%
$l_U / l_0 - 1$	∞	-17%	15%	26%	53%	12%
$l_D / l_0 - 1$	-35%	∞	-53%	7%	-55%	55%
$\sigma_U / \sigma_0 - 1$	-100%	42%	-67%	-50%	-70%	-32%
$\sigma_D / \sigma_0 - 1$	77%	-100%	250%	-57%	170%	-58%

Notes:

Case 1: Riskfree debt and EBV project: $(u, v, d, n) = (1, 1, 0.2, 1)$

Case 2: Riskfree debt and EGV project: $(u, v, d, n) = (1, 0.2, 0.2, 1)$

Case 3: Risky debt and BV project: $(u, v, d, n) = (1.2, 1, 0.4, 0)$

Case 4: Risky debt and GV project: $(u, v, d, n) = (1.6, 1, 0.8, 0)$

Case 5: Risky debt, BV project, and double leverage cycle: $(u, v, d, n) = (1.2, 1, 0.4, 0.2)$

Case 6: Risky debt, GV project, and double leverage cycle: $(u, v, d, n) = (1.6, 1, 0.8, 0.2)$

Common Parameters: $(e, r_f) = (0.2, 0.05)$