Derivative Pricing in the Absence of a Risk Free Rate

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September 9, 2012

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The Risk-Free Interest Rate ?

15 years ago, the LIBOR-Swap curve was treated as a risk-free yield curve at which banks (AA banks) could borrow and lend

This was industry practice up until recent years.

What happened to the risk-free interest rate?

The financial crisis revealed that major banks are not default free, and CDS spreads for these banks have been significantly higher.

LIBOR curves have upward slopes that reflect the increasing credit exposure to banks as the maturity of an uninsured time deposit is increased. 6 month LIBOR is higher than the rate implied by 3 month LIBOR and quotes for in 3 months for 3 month LIBOR FRA's. Banks now incorporate the cost of tenor into swap valuation

Banks are no longer viewed as default-free and the market does not treat LIBOR-Swap rates as rates appropriate for discounting default-free cashflows.

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Real and Nominal Risk-Free Interest Rates

Real and Nominal Risk-Free Interest Rates - Do they exist?

The Real Interest Rate: is an interest rate denominated in real consumption units. It is a concept in economic models. There are market instruments with bond payoffs linked to consumption price indexes: inflation linked bonds and indexed deposits. Real yields on inflation linked bonds serve as approximations for the real interest rate.

Nominal Risk-Free Interest Rates:

Historically, we have treated the sovereigns behind the major currencies of the world as default-free. Hence, their Treasury curves have been viewed as default-free nominal interest rates in their respective currencies. The Euro is a notable exception, where a group of nations agreed to give up their sovereign power to print money, and their Treasury curves trade with credit spread differentials.

Sovereigns are able to borrow in their respective currencies at risk-free rates in nominal terms, with the expectation that the government will always be willing and able to print the additional money necessary to pay its debts, and not actually default.

Examples of Markets for Risk-Free Interest Rates

Treasury Markets (in their respective currencies)

Repo

The repurchase agreement (repo) market is an example of a market where individuals and both financial and non-financial institutions can borrow and lend at default-free interest rates. Collateral, haircuts, and daily mark-to-market are used to ensure that borrowers will repay the loans. Most repo is overnight or short dated.

Default-Free Cashflows embedded in Collateralized Swaps

Most inter-bank swaps and derivatives now contain collateral agreements that require daily mark-to-market and daily transfers of cash collateral to cover market values. Clearing houses also require the same mark-to-market and transfers of cash collateral to cover market values, and they require participants to post additional collateral. The daily marking and cash collateral serve to significantly reduce the counterparty credit exposure. The net result is that one can view the cash payments on these derivatives as streams of default-free cashflows.

Sketch of a General Equilibrium Model for Pricing

Pure arbitrage models take the existence of risk-free interest rates as given. If we want to examine the behavior and determination of interest rates in the absence of a pre-specified curve for risk-free interest rates, we need to go deeper into the fundamentals (go under the covers).

The general equilibrium model of **Cox-Ingersoll-Ross** (1985) provides a framework for interest rate determination and derivative pricing.

The CIR model is an expected utility based model, in which interest rates and risk premia depend on the shape of the utility function for a representative agent. In the model, there is a representative agent who makes decisions regarding investment in a portfolio of productive assets by solving a multi-period consumption-investment problem. Equilibrium pricing relationships are derived from the optimizing behavior of the representative agent.

An equilibrium real interest rate (instantaneous) is derived from the condition that borrowing and lending in the model economy must net to zero; effectively, the equilibrium real interest rate is derived from the representative agent being satisfied with a position of no borrowing and lending at this rate.

All prices and interest rates in the model are denominated in real terms, in units of a single consumption good. At the end of their 2nd paper, CIR introduce a stochastic price index to translate from consumption units to a typical pricing numeraire (denominated in a currency). With nominal prices, they derive a nominal risk-free interest rate (instantaneous) and show that the derivative pricing results from the real pricing model carry over to nominal prices. Naturally, the nominal interest rate includes both the real interest rate and expected (consumption) price inflation.

Shadow Risk-Free Interest Rates

In the CIR model, the economic agents do not borrow at term risk-free interest rates, but they can borrow at a risk-free instantaneous rate, which can be achieved with collateral in this diffusion based model. Within this model, one can value a claim that pays one currency unit in all states (default-free) by applying the familiar discounting formula:

$$D(t,T) \;=\; \hat{E}_t\left(\exp\left(-\int_t^T r(s)\,ds\right)\right)$$

In effect, we have shadow risk-free interest rates, and a concept of discounting for risk-free nominal cashflows, even though the agents in the economy cannot borrow at these rates.

Overnight repo is one example of default-free borrowing and lending. Overnight Fed funds and EONIA are rates available to only high quality borrowers. If we assume that this overnight borrowing and lending is also default-free, or nearly default-free, then we have another proxy for a risk-free interest rate. Markets have developed for swaps and forwards on these overnight rates, the OIS market, so that one can easily extend the discount function and build forward rate curves.

Even if the overnight interest rate is not a risk-free rate, but is used for the payment of interest on cash collateral posted for a derivative, it has been shown that this interest rate is the rate that should be used for discounting the cashflows on that derivative contract. (See Johannes and Sundaresan (2007)

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Derivatives Valuation

r - the risk-free rate at which a bank can borrow or lend

r_c - the interest rate paid on the collateral

$X(T)$ - the cashflow or payoff on a derivative at time T, then

$$V(0) = \hat{E}_0\left(e^{-\int_0^T r(u)du} g(X(T)) \right) + \hat{E}_0\left(\int_0^T e^{-\int_0^s r(u)du} V(s)\,[r(s) - r_c(s)]\, ds \right)$$

$$= \hat{E}_0\left(e^{-\int_0^T [r(u)-(r(u)-r_c(u)]du} g(X(T)) \right) = \hat{E}_0\left(e^{-\int_0^T r_c(u)du} g(X(T)) \right)$$

In words, if the collateral amount each day in the future is equal to the value of the trade and the overnight collateral rate is paid on collateral every day, then discounting the trade cashflows with the collateral rate produces the correct valuation, even though the collateral rate may differ from the risk-free rate. [see Johannes & Sundaresan (2007)] The second term above for the collateral cashflow represents the difference between the opportunity cost for risk-free funds versus the rate paid or received on the collateral; it works like a continuous dividend yield.

Recap

Good news:

We have all the necessary ingredients for applying derivative pricing models, including a rate for discounting default-free cashflows. We can build up a term structure for forward rates and discounting functions. The major change is that we should not be using the LIBOR-Swap curve for discounting. For swap contracts, we still need forward rates for the different LIBORs.

Current market practice:

1) build a discounting curve from OIS swaps and

2) build forward rate curves for forward LIBOR,

with separate curves for 3m, 1m, 6m and 12m LIBOR forwards.

Derivative pricing results for stochastic interest rates models are the same, with OIS discounting rates treated as stochastic. The same familiar results hold for both short rate models and forward rate models. Some additional results are necessary for simulating a stochastic basis between LIBOR and OIS. All of this is possible with a discount curve based on shadow risk-free interest rates

Comments and Observations regarding model simulations (deterministic vs. stochastic LIBOR-OIS basis)

Extensions to a Multi-Currency Model

One can extend the CIR model by introducing multiple price indexes, corresponding to multiple pricing numeraires. In this sense, the model is one of economic agents making real consumption and investment decisions in a multi-currency world. In this case, the currency or the price index serves only as a numeraire for quoting prices and transacting. A more realistic, but more complicated, approach is to introduce money for each currency. For examples with applications to asset pricing, see Bakshi and Chen (1997) and Stulz (1987).

In the multi-currency model, the FX rate is the price of one currency denominated in another currency. It is determined by the ratio of the stochastic price indexes. Define S_{ij} to be currency *i* price of currency *j*, expressed as units of currency *i* per unit of currency *j* :

$$S_{ij} = \frac{p_i}{p_j}$$

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The Multi-Currency Model

Starting from diffusion processes for the price indexes in each currency,

$$\frac{dp_i}{p_i} = \mu_i^*(X, Y, p_i)\, dt + \sigma_i^*(X, Y, p_i)\, dz_i^*$$

we can derive the stochastic differential for the FX rate.

$$dS_{ij} = S_{ij}\,[\,(r_i(X, Y, p_i) - r_j(X, Y, p_j)dt + \sigma_i^*(X, Y, p_i)dz_i^* - \sigma_j^*(X, Y, p_j)\, dz_j^*\,]$$

Here *Y* represents a vector of stochastic state variables that determine the mean and variances for the real asset returns, as well as the real interest rate, and may play a role in the processes for the consumption price indexes. *X* represent a vector of stochastic variables that determine means and variances for the price index processes, that are independent of stochastic processes for the real side of the economy.

In the model, the risk premia that are applied under the different pricing measures are linked as follows:

$$\phi_i(Y, t) = \phi_j(Y, t) - \operatorname{cov}\left(\frac{dS_{ij}}{S_{ij}}, dY\right) .$$

This is the quanto adjustment that needs to be applied in multi-currency models. An obvious example is the simulation of stochastic volatility correlated with the FX rate.

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Foreign Rates

Simulating the foreign interest rates under the domestic pricing measure. Does the following relationship hold?

$$\hat{E}_{it}\left(\exp\left(-\int_t^T r_j(s)\,ds\right)\right) = \hat{E}_{jt}\left(\exp\left(-\int_t^T r_j(s)\,ds\right)\right) \equiv D_j(t,T)\,.$$

The answer in the model depends on the quanto adjustment. If the state variables that drive interest rates are not correlated with exchange rates, then the answer is Yes. Otherwise, the answer is No because the foreign interest rate follows a different risk neutral process under the domestic currency pricing measure. We shall see that the answer is No in current markets.

$$\hat{E}_{it}\left(\exp\left(-\int_t^T r_j(s)\,ds\right)\right) \neq \hat{E}_{jt}\left(\exp\left(-\int_t^T r_j(s)\,ds\right)\right) \equiv D_j(t,T)\,.$$

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FX Forward Pricing

FX Forward Pricing

In an FX forward contract, there is a future cashflow determined by the difference between the future FX rate and the fixed FX forward rate. In the model, we have the following pricing relationship:

$$\hat{E}_{it}\left(\exp\left(-\int_t^T r_i(s)\, ds \right) \; [\; F_{ij}(t,T) - S_{ij}(T) \;] \right) \;=\; 0 \;\; .$$

The discounted forward gives us the domestic currency valuation of a default-free cashflow in the foreign currency:

$$D_i(t,T)\; F_{ij}(t,T) = \hat{E}_{it}\left(\exp\left(-\int_t^T r_i(s)\, ds \right) S_{ij}(T) \right)$$

$$= S_{ij}(t)\; \hat{E}_{it}\Bigg(\exp\Bigg[-\int_t^T r_j(s)\, ds$$

$$- \tfrac{1}{2} \int_t^T (\sigma_i^{*\,2}(Y(s)) + \sigma_j^{*\,2}(Y(s) + (\beta_i - \beta_j)^2\, \sigma_0^{*\,2}(Y(s)))\, ds$$

$$+ \int_t^T (\sigma_i^{*}(Y(s))\; dz_i^{*}(s) - \sigma_j^{*}(Y(s))\; dz_j^{*}(s) + (\beta_i - \beta_j)\; \sigma_0^{*}(Y(s))\; dz_0^{*}(s)) \Bigg] \Bigg)$$

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Covered Interest Rate Parity

What about covered interest rate parity (covered interest arbitrage)?

$$D_i(t,T)\, F_{ij}(t,T) \;=\; S_{ij}(t)\, D_{ij}^{*}(t,T)\,.$$

In the model, the following condition ensures that covered interest rate parity is satisfied: no correlation between the variables that determine the foreign interest rate and the Brownian motions in the FX process. This condition also ensures that the discounting function for the foreign interest rate is the same under both the foreign and the domestic pricing measures.

Alternatively, if there are economic agents who can borrow and lend at term risk-free interest rates, this pricing relationship would hold, otherwise there would be arbitrage opportunities (for the agents who borrow at risk-free rates) . In the model, we have assumed that the economic agents are not risk-free, and can only borrow at risk-free rates if sufficient collateral is posted. If banks can borrow unsecured at term rates near the shadow risk-free rates, then this relationship might work as an approximation.

Covered interest rate parity was approximately accurate prior to the financial crisis. Since the onset of the financial crisis, FX forward markets have deviated significantly from this relationship.

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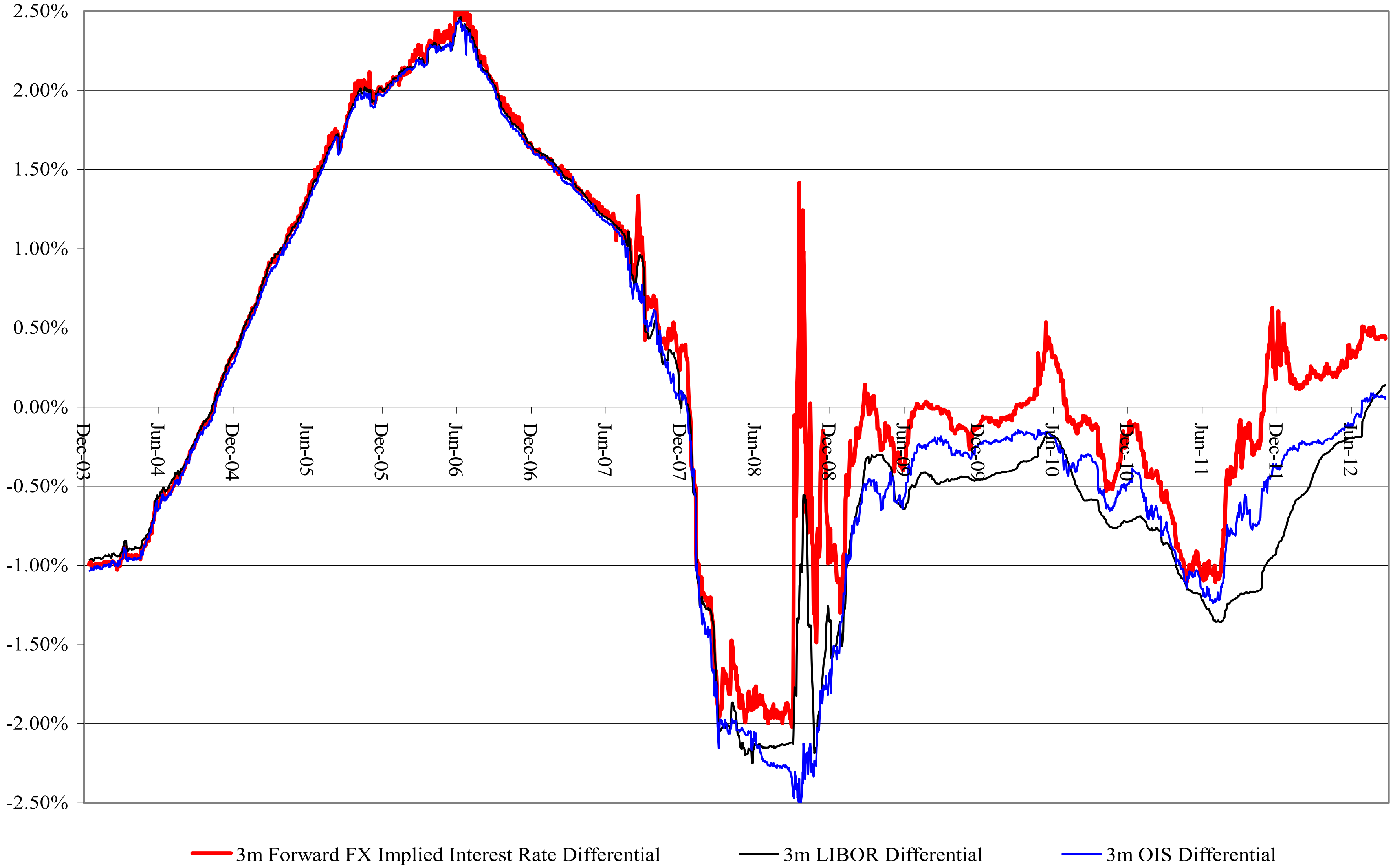
3m Forward FX Implied Interest Rate Differentials, USD-EUR



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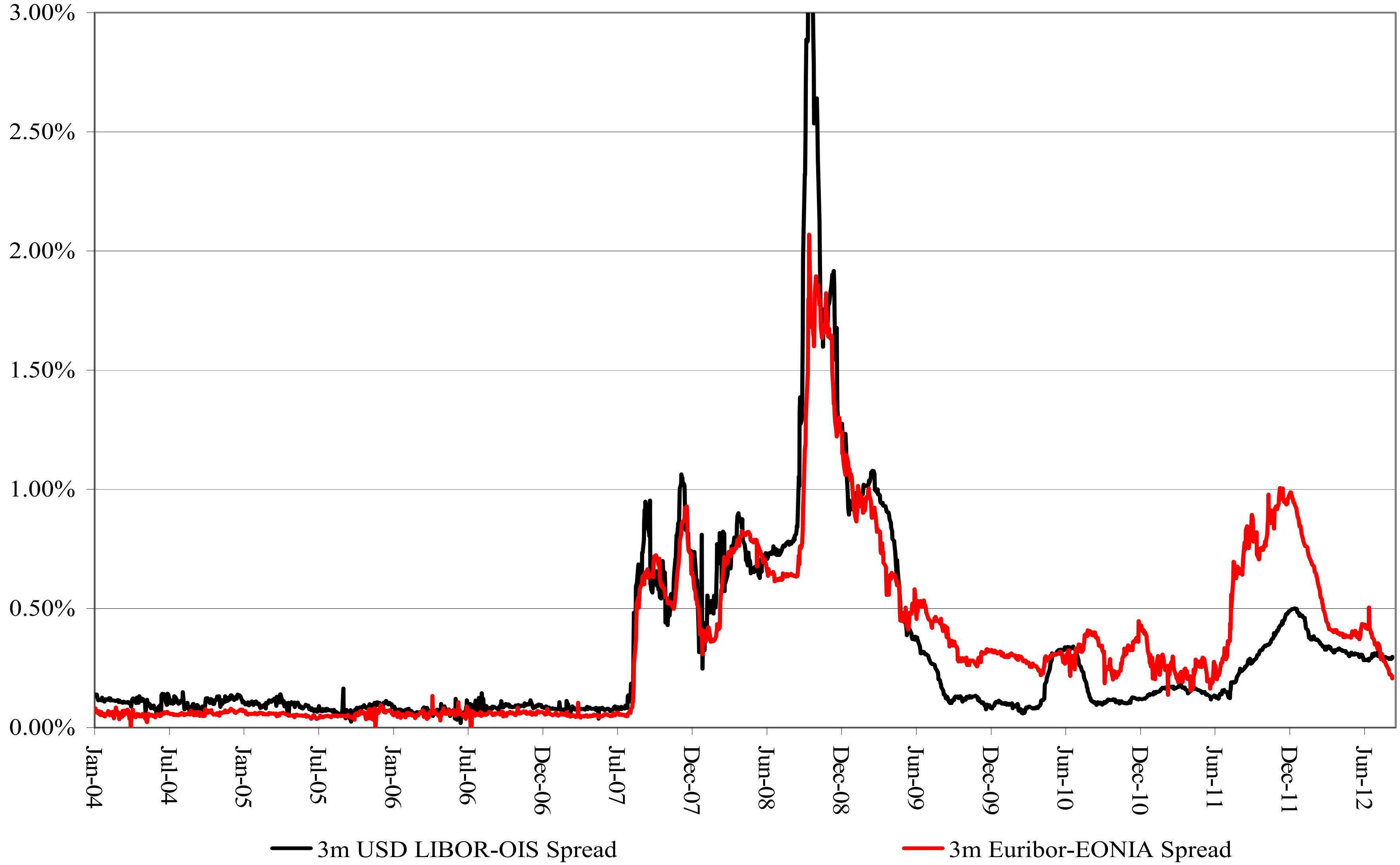
Historical Graph of LIBOR-OIS Spreads



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S_{ij}

Multi-Currency Yield Curve Calibrations

The following slides contain yield curve calibrations for USD-Euro, USD-GBP, and USD-JPY.

The calibrations were performed with USD as the pricing measure.

The input data for the calibrations include:

- USD OIS Swap quotes
- USD LIBOR FRA's and Swap quotes
- OIS, LIBOR FRA, and LIBOR Swap quotes in the foreign currency
- FX Spot and Forwards (out to 10 years)
- Cross-currency LIBOR Swap quotes

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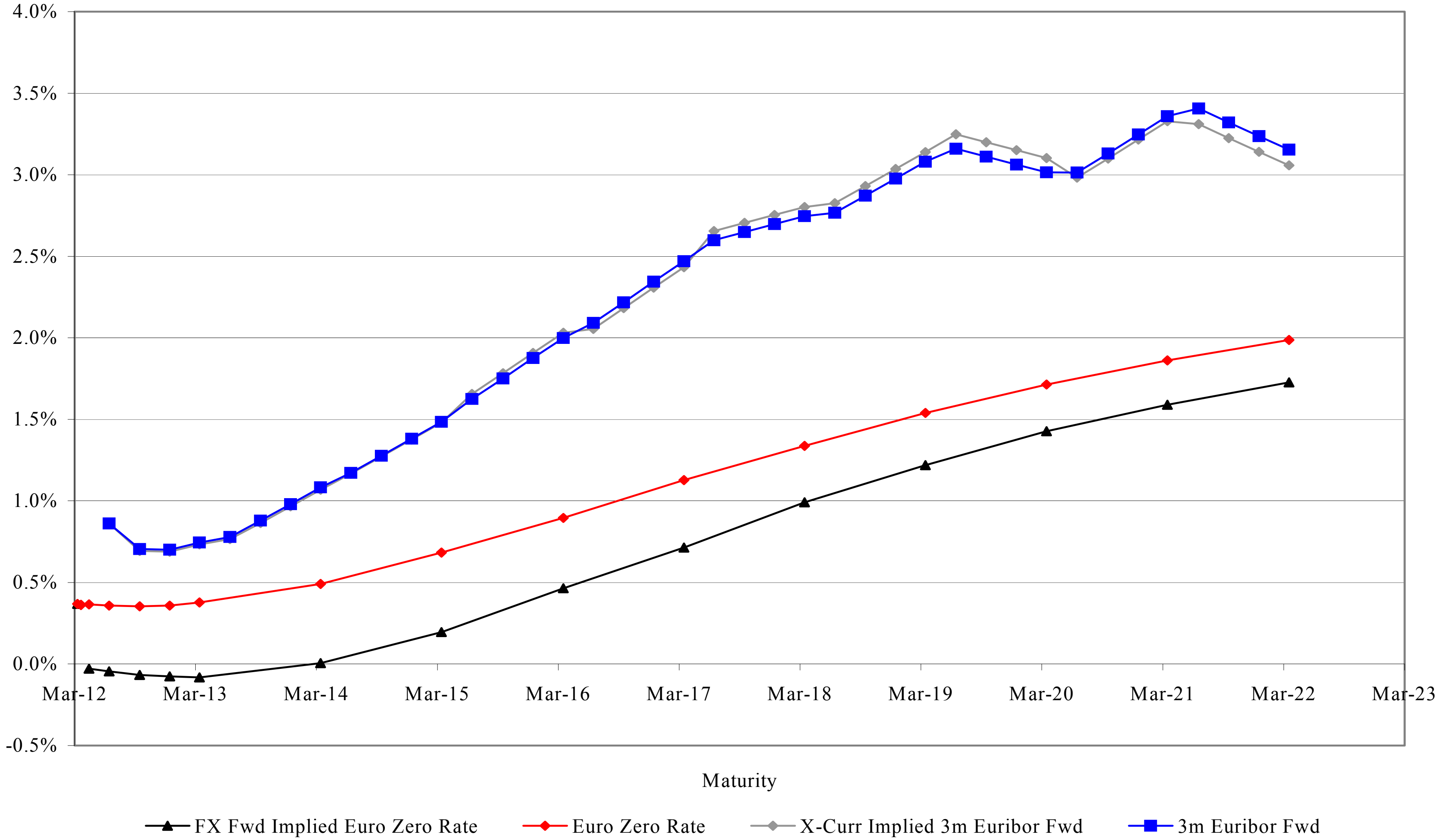
USD-Euro Multi-Currency Yield Curves, Mar. 15, 2012



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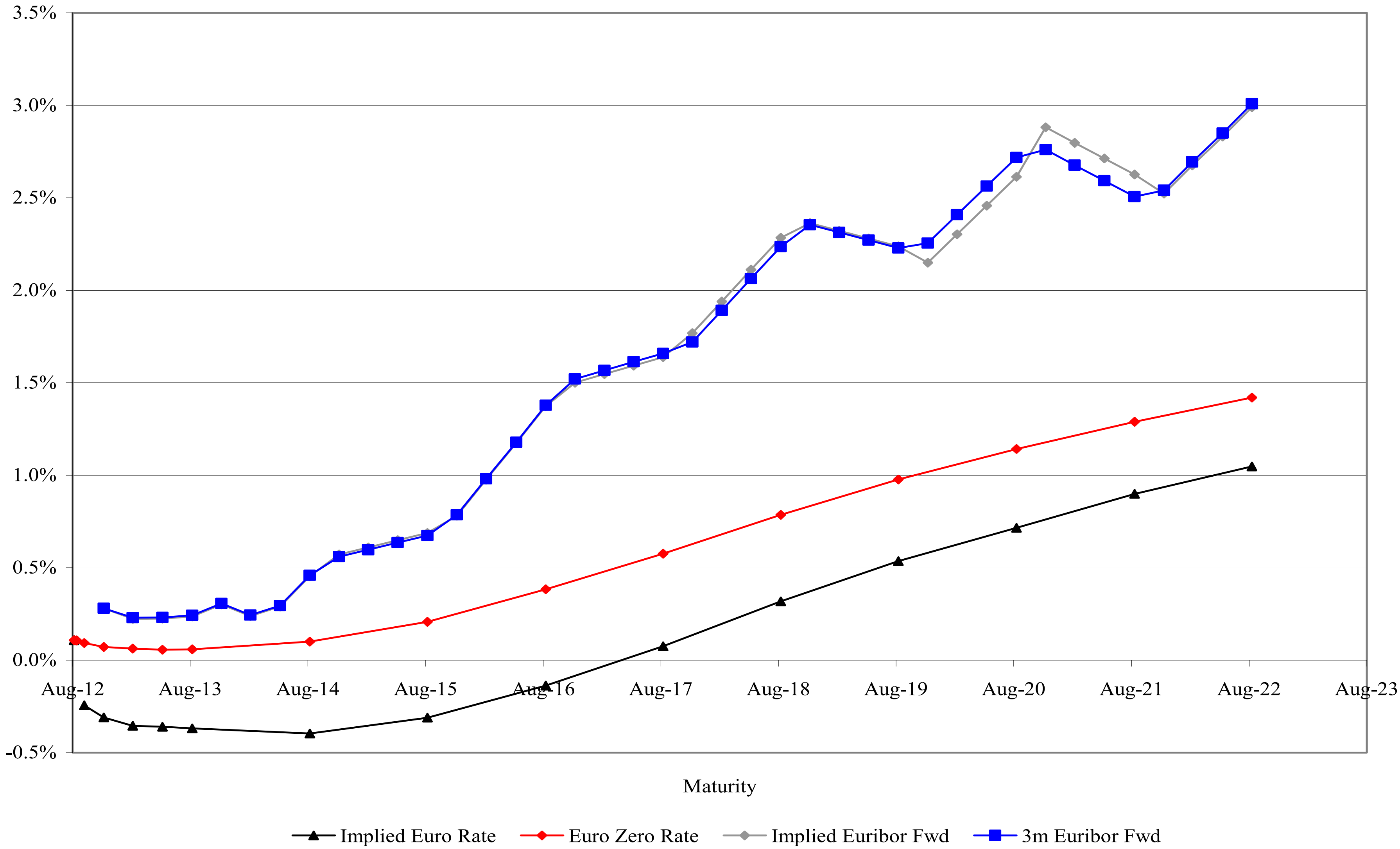
USD-Euro Multi-Currency Yield Curves, Aug. 31, 2012



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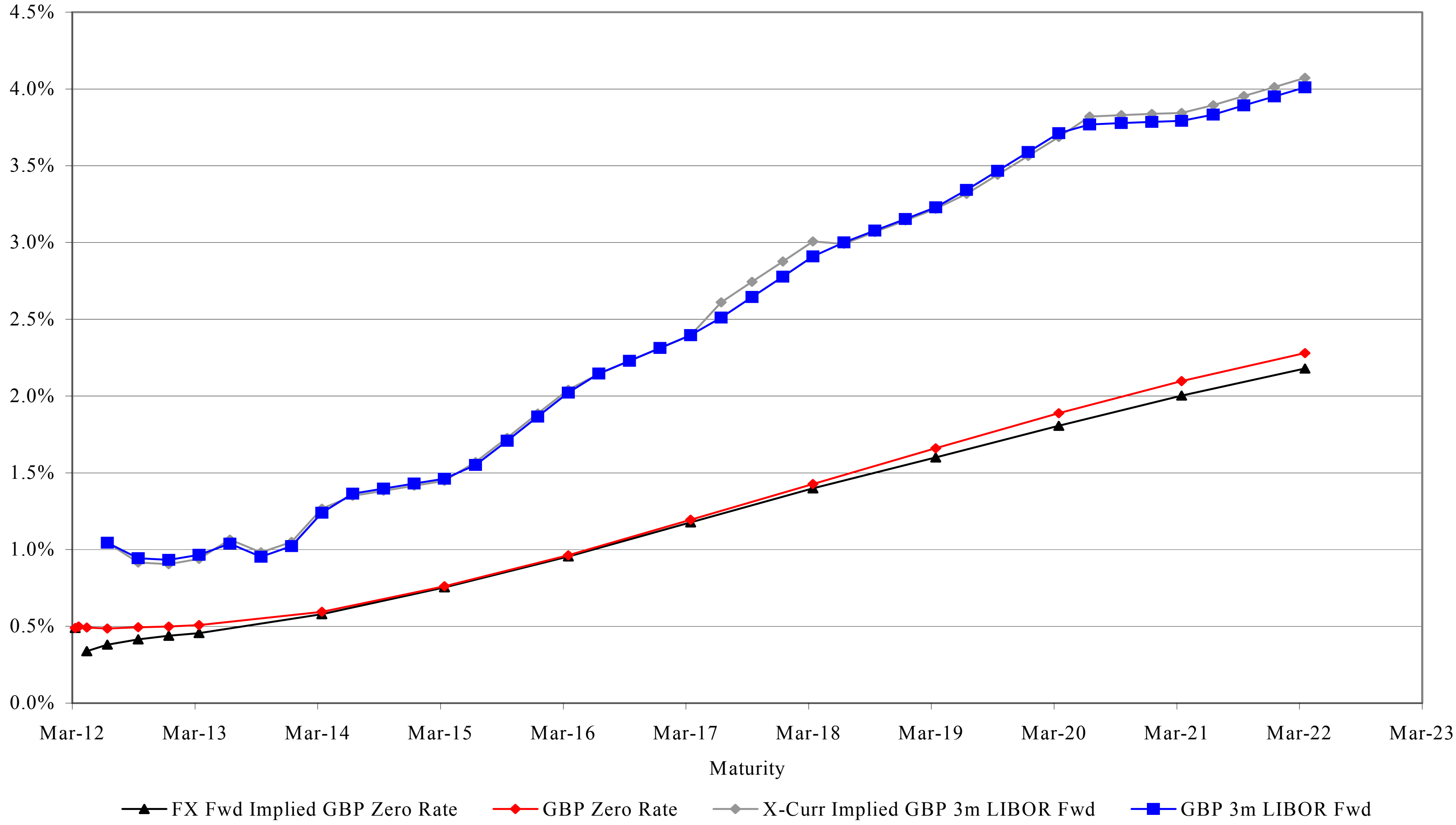
USD-GBP Multi-Currency Yield Curves, Mar. 15, 2012



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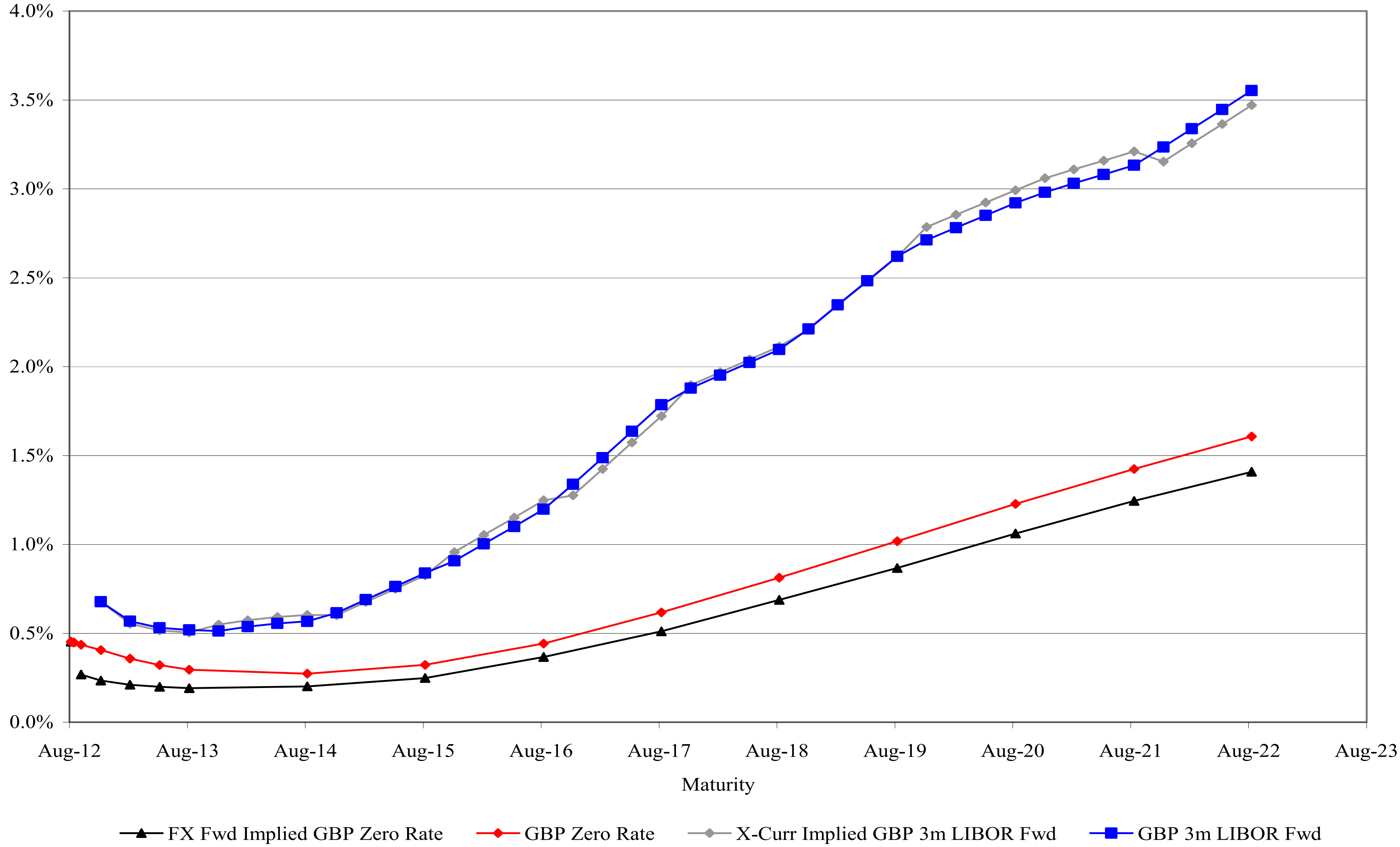
USD-GBP Multi-Currency Yield Curves, Aug. 31, 2012



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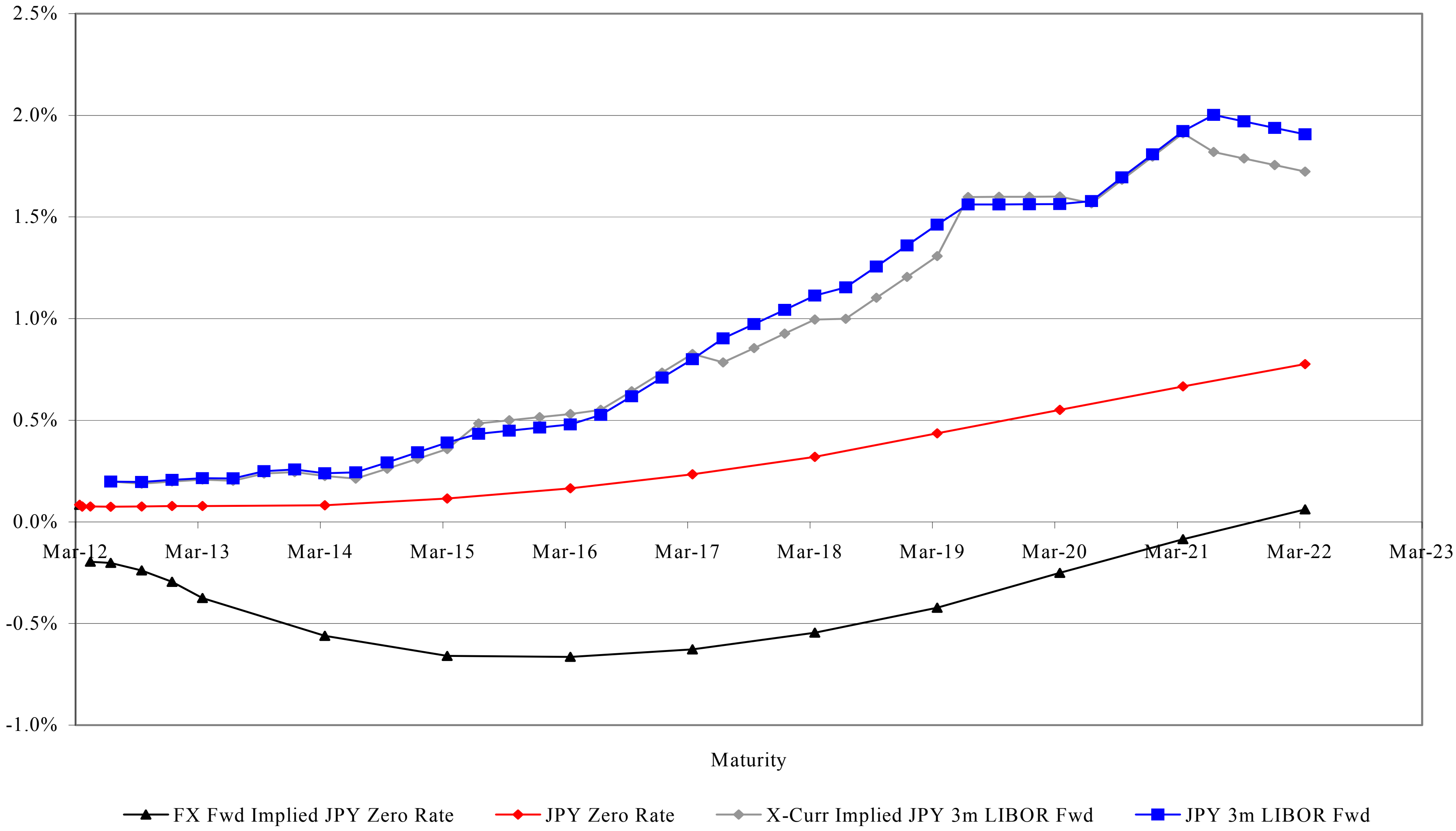
USD-JPY Multi-Currency Yield Curves, Mar. 16, 2012



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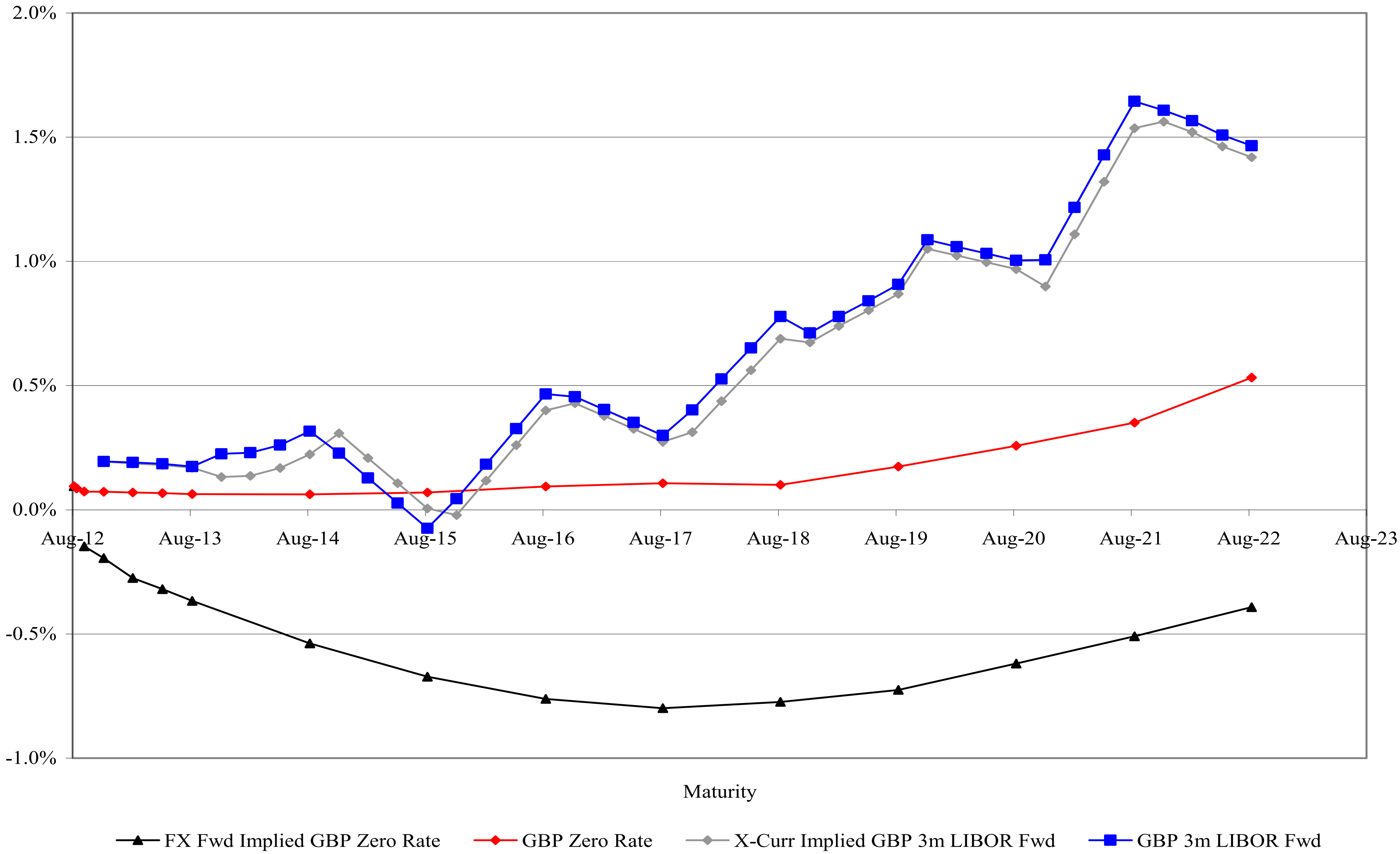
USD-JPY Multi-Currency Yield Curves, Aug. 31, 2012



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Collateral Arrangements and Clearing Houses

Collateral and counterparty exposure are important. Hence the recent focus on models for Credit Valuation Adjustments (CVA) and attempts to incorporate collateral into pricing and valuation.

Almost all derivative trades between banks are now collateralized. These include interest rate swaps, cross currency swaps, FX forwards, FX options, interest rate options, and OTC equity options. The collateral serves to eliminate most of the credit risk exposure between derivative counterparties.

The market values for these derivatives are calculated daily, and collateral must be posted to cover the net MV difference between two banks. If the net MV of Bank B's derivatives with Bank A is $20 MM, then Bank A will post $20 MM in cash collateral with Bank B. Bank B is obligated to pay an overnight interest rate on this cash collateral. The overnight rates are typically the effective Federal Funds rate for USD, TOIS for CHF, EONIA for Euro, SONIA for GBP, …

Many of the interest rate swaps between banks are now cleared through the London Clearing House. LCH requires collateral to cover the MV of the swap and they pay the overnight interest rate in that currency on the cash collateral.

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Derivatives Valuation with Collateral in Another Currency

Consider a case where we have a derivative payoff in a domestic currency, but the collateral is posted in a foreign currency, with the foreign O/N rate paid on balances.

Case 1: The previous results cited indicate that we need to discount at the foreign interest rate; hence we shift to the foreign pricing measure and price the derivative under the foreign currency pricing measure with the appropriate quanto adjustments. This implies that both counterparties agree to use the foreign pricing measure valuation, denoted $V^*(0)$, to determine collateral balances.

Case 2: Suppose that instead we use the domestic currency valuation, $V(0)$, to determine collateral balances. Given the basis in FX forwards and cross-currency swaps, $V(0) \neq V^*(0)\, S_{ij}(0)$.

Analysis for the valuation of the collateral cashflow under the domestic currency pricing measure (Equity derivative valued in HKD, with collateral posted and managed in USD, resettled daily)

r_i - the domestic risk-free interest rate, instantaneous or overnight

r_j - the foreign risk-free interest rate, instantaneous or overnight

$X(T)$ - the cashflow or payoff on a derivative at time T, denominated in domestic currency

For the period 0 to T, use N periods of length Δt

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Derivatives Valuation with Collateral in Another Currency

The opportunity cost for the collateral balance from $j\Delta t$ to $(j+1)\Delta t$ is $V(j\Delta t)\,[1+r_i(j\Delta t)\Delta t]$

The daily payout for the collateral (negative cashflow when $V(0)$ is positive) is

$$-V(k\Delta t)\,[1+r_j(k\Delta t)\Delta t]\frac{S_{ij}((k+1)\Delta t)}{S_{ij}(k\Delta t)} = -V(k\Delta t)\left[1+r_j(k\Delta t)\Delta t+\frac{S_{ij}((k+1)\Delta t)-S_{ij}(k\Delta t)}{S_{ij}(k\Delta t)}+r_j(k\Delta t)\Delta t\left(\frac{S_{ij}((k+1)\Delta t)-S_{ij}(k\Delta t)}{S_{ij}(k\Delta t)}\right)\right]$$

The valuation under the domestic currency pricing measure is

$$V(0)=\hat{E}_0\left(e^{-\int_0^T r_i(u)du}\,g(X(T))\right)+\hat{E}_0\left(\sum_{k=0}^{N-1}\exp\left(-\sum\nolimits_{m=0}^{k} r_i(m\Delta t)\Delta t\right)V(j\Delta t)\left[r_i(k\Delta t)\Delta t-r_j(k\Delta t)\Delta t-\left(\frac{S_{ij}((k+1)\Delta t)-S_{ij}(k\Delta t)}{S_{ij}(k\Delta t)}\right)-O(\Delta t^2)\right]\right)$$

The expectation in the second term can be evaluated on a period by period basis. When we apply the martingale property for the FX rate, the interest rate differential and the change in the FX rate cancel, leaving only terms that go to zero with Δt^2 . We are left with the standard valuation model under the domestic currency pricing measure.

$$V(0)=\hat{E}_0\left(e^{-\int_0^T r_i(u)du}\,g(X(T))\right)$$

Conclusion: if the collateral balance is managed to the domestic currency valuation, no further adjustments are necessary in the domestic currency valuation. If the collateral balance is managed to the foreign currency valuation, then pricing under the foreign currency pricing measure, with quanto adjustments, is also OK. Both methods work, if applied consistency. Pricing under the domestic currency will be easier to implement.

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Summary – Risk-Free Rates & Derivative Pricing

In the absence of borrowing at risk-free interest rates, there are shadow risk-free rates, which can be used to value default-free cashflows on derivative contracts.

Extra care is necessary when running derivative pricing models with stochastic interest rates and stochastic volatility for FX rates

- simulation of the foreign interest rate under the domestic pricing measure

- stochastic LIBOR-OIS spreads

- stochastic FX forward basis and stochastic cross-currency basis

If you ignore some of the subtleties in dynamic asset pricing, you run the risk of making mistakes. Example: a poorly designed multi-currency model will produce simulations that do not reprice the FX forwards and European options that were used in the model calibration.

Switching the currency of the collateral: is it much ado about nothing?

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