Default Option and Optimal Capital Structure in Real Estate Investment

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This article investigates the determinants of optimal capital structure in real estate investment in a real options framework where an investor incurs transaction costs when purchasing a property through applying for a mortgage loan. We assume that an investor chooses a date at which to purchase the property, and at that date, decides a loan-to-value ratio, which balances the tax shield benefit and the transaction costs. After the purchase, the investor has the option to default if the property value falls significantly. As this option becomes more valuable either from greater uncertainty in housing price inflation or longer duration of the loan, the investor will gain more through purchasing earlier and reducing debt financing.

Key words: default; optimal capital structure; real estate investment; real options; transaction costs

JEL: G13, G31, G32.

I. Introduction

This article investigates the investment and financing decisions of a real estate investor who considers acquiring a residential or self-used commercial property through debt financing. Some articles have considered the optimal financing decision of an investor who intends to purchase an income-generating property. See, for example, Cannaday and Yang (1995, 1996), Gau and Wang (1990), and McDonald (1999).[1] All of these articles do not allow the investor to have the option to default nor to delay purchasing the property. This article significantly differs from them by allowing the investor to have these two options.

This article, which belongs to the burgeoning literature that applies the real options approach to investment (Dixit and Pindyck, 1994), assumes that an investor chooses both an optimal timing and a loan-to-value (LTV) ratio to maximize the expected net present value from purchasing a property. We start from the observation that an investor must pay more sunk costs in the form of increasing mortgage rates or commission fees if increasing the LTV ratio.[2] The interaction of these sunk costs and the stochastic evolution of the property value confers on the investor an option value to delay purchasing the property. Consequently, the investor will not purchase until the value of the property reaches a threshold level. At that threshold level, the investor also decides a LTV ratio that involves a tradeoff between the sunk costs of debt financing and the tax deductible benefit from interest payments and/or capital depreciation.[3] After signing the mortgage contract, the investor has the option to default should the value of the property fall significantly.[4]

An investor certainly benefits from the option to default. However, in general we are not sure how it affects the investor's incentives to borrow and invest because of the conflicting effects arising from the two channels as follows: (i) given the LTV ratio, the investor will

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[1] Ever since the seminal paper by Modigliani and Miller (1958), the determinants of corporate borrowing have been a heated topic in the corporate finance literature. See, for example, the survey paper by Harris and Raviv (1991), and Myers (2003). This topic receives little attention, however, in the real estate investment literature. See the discussions in Gau and Wang (1990) and Clauretie and Sirmans (2006, chapter 15).

[2] We consider the sunk costs at the date when an investor applies for a mortgage loan rather than when the investor defaults as addressed in Green, Rosenblatt and Yao (2010).

[3] This tradeoff significantly differs from that addressed in the finance literature, which also allows the tax advantages of borrowing, but considers the costs associated with either financial distress, or the conflict of interest between equityholders and debtholders. See, for example, Harris and Raviv (1991) and Myers (2003).

[4] This article also differs from Gau and Wang (1990) and McDonald (1999), as these two studies assume that an investor bears the cost associated with bankruptcy (Stiglitz, 1972) when failing to pay off debt obligations. This article, however, abstracts from the bankruptcy cost.

purchase a property earlier when having this option value, which, in turn, will induce the investor to borrow less because the investor purchases the property at a less favorable state of nature;[5] and (ii) given the state of nature, an investor with this option will borrow more, which, in turn, will induce the investor to delay purchasing the property because waiting then becomes more valuable. By employing plausible parameter values, we find that each effect derived from channel (i) more than offsets its counterpart derived from channel (ii) such that the existence of the option to default induces the investor to purchase sooner and borrow less. Parallel with this finding, an investor whose option value to default becomes more valuable, either because housing price inflation becomes more volatile or because the mortgage loan is long-lived, will gain more through purchasing sooner and borrowing less.

This article provides some testable implications regarding the determinants of mortgage default, which have received wide attention in the literature. Foster and Van Order (1984) first proposed mortgage default as a put option. Kau, Keenan and Kim (1993) further shows that, given that an investor has the option to default, the investor will not default the mortgage loan even when the housing value just falls short of the loan balance.[6] Deng, Quigley and Van Order (2000) indicate that heterogeneity among households can explain the default probability. Although the literature (see, e.g., Quercia and Stegman, 1992) consistently finds that the initial LTV as the main characteristic affecting the default decision, yet Kau, Keenan and Yildirim (2009) claim that in the absence of any taxation, the initial LTV ratio will not affect the default probability. Nevertheless, we demonstrate that the initial LTV matters if an investor enjoys tax shield benefit of debt financing.

The remaining sections are organized as follows. We first present the basic assumptions of the model. We then analyze the determinants of default timing, given that the investor has already signed the mortgage loan. The ability for the investor to default, in turn, affects the investor's choices of investment and debt financing decisions. Most of our theoretical predictions regarding these two choices, however, are indeterminate, and thus we employ plausible

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[5] This resembles the effect stated in the two-period model of Abel, Dixit, Eberly and Pindyck (1996), which shows that an investor who acquires an option value to abandon will choose a larger scale of an investment project.

[6] Hilliard, Kau and Slawson (1998) and Schwartz and Torous (1992) investigate how the interaction of default option and the prepayment option affects the mortgage loan valuation. We abstract from the prepayment option because we assume that the interest rate is constant over time, which is especially plausible for an investor who applies for a commercial mortgage loan and thus must pay substantial penalties if paying off the loan before the maturity date (Cannaday and Yang, 1996).

parameters in order to carry out some numerical comparative-statics testing in the following section. The last section concludes and offers suggestions for future research.

II. The Model

Consider an investor who decides the date at which to purchase a property, as well as the percentage of debt to finance this purchase. The value of the property at time s, $H(s)$, evolves as a geometric Brownian motion as

$$\frac{dH(s)}{H(s)} = (\mu - \lambda)ds + \sigma d\Omega(s), \tag{1}$$

where $d\Omega(s)$ is the increment of a standard Wiener process, μ is the total expected return from holding the property, λ is the service flow rate, and σ is the instantaneous volatility of housing price inflation. The property in consideration may be a self-owned residential or commercial property, and thus we can interpret the service flow rate as the imputed rental rate. Suppose that we start from $t = 0$, and denote $H(0)$ as the initial value of the property. At the initial date, the investor purchases the property and chooses a LTV ratio, denoted as M, such that the investor carries a mortgage loan with an initial balance $MH(0)$, while injecting an initial equity equal to $(1-M)H(0)$. We assume that the mortgage interest rate is equal to a constant r and that the mortgage pays interest only before the maturity date T. We also assume that all agents are risk neutral, and employ the constant interest rate as their discount rate.[7] Thus, at each instant after purchasing a property, the investor pays interest equal to $rMH(0)$ to the loan provider.

Significantly different from previous research, we assume that an investor suffers a sunk cost from borrowing equal to fM^{ε}, where $f > 0$, and $\varepsilon > 1$. This sunk cost comes form two sources. First, the commission fee charged by a loan provider usually increases with the LTV ratio. For example, a loan origination fee is typically proportional to the amount of borrowing and the proceeds of the loan discount point are exactly proportional to the amount of borrowing. Second, we assume that the mortgage rate is fixed. However, as Cannaday and Young (1996) suggest, an investor typically is charged at a higher mortgage rate if increasing the LTV ratio. Therefore, our specification also captures the additional borrowing cost that is not accounted for by the fixed mortgage rate. Combining these two sources indicate that the sunk cost is

[7] We can generalize our model to the case of risk aversion in the manner of Cox and Ross (1976). Our result, however, will be the same regardless of whether we consider a risk-neutral, or a risk-averse, environment.

approximately an increasing convex function of the LTV ratio, i.e., $\varepsilon > 1$. That is, as the LTV ratio increases, the sunk cost of debt financing increases more than proportionally.[8] The total cost paid by the investor at the instant of purchase is thus given by

$$C(H(0), M) = (1-M)H(0) + fM^{\varepsilon}. \tag{2}$$

In contrast to the literature on optimal capital structure of real estate investment, we allow the investor to have the option to default. Following Leland (1994), we assume that the investor will default the loan when the property value declines to a value, denoted as $H_b(s)$ at time s, such that the investment value falls to zero. Denote $V(H(t),t)$ as the investment value at time t. Applying risk-neutral valuation, and following Leland and Tuft (1996) yields

$$V(H(t),t) = \int_t^T e^{-r(s-t)}[\lambda H(s) - (1-\tau)rMH(0) + \frac{\tau\delta H(0)}{n}][1 - L(H(t), H_b(s), s)]ds \tag{3}$$

$$+e^{-r(T-t)}E_t[1 - L(H(t), H_b(T), T)]ATER(T),$$

where $L(H(t), H_b(s), s)$ is the cumulative distribution function of the first passage time to default at time s starting from time t, δ is the proportion of the property that is depreciable capital (that is, not land), n is the length of the depreciation period (39 years for commercial real estate in the U.S.), τ is the income tax rate, E_t is the expectation operator applied at time t, $ATER(T)$ is after-tax equity reversion from selling the property at time T. Note that we must set δ equal to zero for an investor who applies for a residential mortgage loan, given that the investor is not allowed to enjoy capital depreciation for tax purposes. Equation (3) indicates that an investor expects to receive proceeds provided that default does not occur, as indicated by the two terms on the right-hand side: the first term is the sum of the imputed rental rate, the after-tax interest payment, and the expected benefit from tax deductions of depreciation allowance. The second term is the expected present value of $ATER(T)$, where[9]

$$ATER(T) = H(T) - MH(0) - \tau[H(T) - H(0) + \frac{\delta T}{n}H(0)]. \tag{4}$$

[8] The sunk cost may also include the non-monetary cost that mainly reflects the time spent in negotiating with the loan provider, which is also likely to increase with the LTV ratio.
[9] Here we assume that T is smaller than n. If T is larger than n, then we must replace T by n.

In Equation (4), $H(T)$ is the revenue the investor derives from selling the property at date T, while $MH(0)$ and $\tau[H(T)-H(0)+\delta TH(0)/n]$ are respectively the loan balance and the payment on capital gains tax.

Following the traditional literature on optimal capital structure of real estate investment such as Cannaday and Yang (1996), Gau and Wang (1990), and McDonald (1999), we assume that the investor acts in his/her interest and thus chooses an appropriate timing and a LTV ratio so as to maximize the investment value $V(\cdot)$ in Equation (3), net of the investment cost $C(\cdot)$ in Equation (2). [10]

Given that the investor incurs sunk costs in purchasing a property through debt financing and that the property value is stochastic, we are thus unable to find a non-stochastic timing of investment (Dixit and Pindyck, 1994, p.139). Instead, the investment rule takes the form where the investor will not purchase the property unless the property value reaches a critical level, denoted by H^*. At that instant, the investor will choose a LTV ratio, denoted by M^*. After the purchase, the investor will default whenever the property value declines to $H_b(s)$ at instant s.

Using risk-neutral valuation and applying Ito's lemma yields the differential equation applying to $V(H(t),t)$ as given by:

$$\frac{\sigma^2}{2}H(t)^2\frac{\partial^2 V(\cdot)}{\partial H(t)^2}+(r-\lambda)H(t)\frac{\partial V(\cdot)}{\partial H(t)}+\frac{\partial V(\cdot)}{\partial t}+\lambda H(t)+[\frac{\delta\tau}{n}-(1-\tau)rM]H(0)=rV(\cdot). \tag{5}$$

Equation (5) has an intuitive interpretation. If we treat $V(\cdot)$ as an asset value, then the expected capital gain of the investment (the sum of the first three terms on the left-hand side) plus the dividend (the remaining two terms on the left-hand side) must be equal to the return required by the investor (the term on the right-hand side). Furthermore, two boundary conditions applied to $V(\cdot)$ are given as follows:

$$V(H(T),T)=H(T)-MH(0)-\tau[H(T)-H(0)+\frac{\delta T}{n}H(0)], \tag{6}$$

and

$$V(H_b(s),s)=0. \tag{7}$$

[10] In the standard finance literature on optimal capital structure (see, e.g., Myers, 2003), firms are assumed to choose debt levels to maximize total firm value. This contrasts with the standard literature on real estate investment where an investor is assumed to choose debt levels to maximize equity value.

Equation (6) indicates that the investment value is equal to after-tax equity reversion from selling the property when the mortgage loan matures. Equation (7) indicates that for any instant prior to maturity, i.e., $0 < s < T$, the investor will default once the investment value just falls to zero.

In order to investigate the investor's financing decision, we define $G = \partial V(\cdot)/\partial M$. Differentiating Equation (5) term by term with respect to M yields

$$\frac{\sigma^2}{2}H(t)^2\frac{\partial^2 G(\cdot)}{\partial H(t)^2}+(r-\lambda)H(t)\frac{\partial G(\cdot)}{\partial H(t)}+\frac{\partial G(\cdot)}{\partial t}-(1-\tau)rH(0)=rG(\cdot)\,. \tag{8}$$

The choice of the initial LTV ratio, M^*, is derived by setting the derivative of the net value of investment, $V(H(0),T)-C(H(0),M)$, with respect to M equal to zero. Evaluating the result at $H(0)=H^*$ and $M=M^*$ yields

$$G(H^*,0)+H^*-f\varepsilon M^{*\varepsilon-1}=0\,. \tag{9}$$

An investor is free to choose a date at which to purchase a property such that the investor's option value of waiting will be independent of the calendar date. Thus, we can denote this value as $F_1(H(t))$, which satisfies the ordinary differential equation given by

$$\frac{\sigma^2}{2}H(t)^2\frac{d^2F_1(\cdot)}{dH(t)^2}+(r-\lambda)H(t)\frac{dF_1(\cdot)}{dH(t)}=rF_1(\cdot)\,. \tag{10}$$

The choice of the investment timing is characterized by both the value-matching condition

$$F_1(H^*)=V(H^*,T)-C(H^*,M^*)\,, \tag{11}$$

and the smooth-pasting condition

$$\frac{dF_1(H^*)}{dH(0)}=\frac{\partial V(H^*,T)}{\partial H(0)}-\frac{\partial C(H^*,M^*)}{\partial H(0)}\,, \tag{12}$$

where we have evaluated Equations (11) and (12) at $H(0)=H^*$ and $M=M^*$.

Appendix A shows the hypothetical case as considered in Kau et al. (2009) where an investor applies for a perpetual mortgage loan. The analysis for this case is quite simple because the investor will not default unless the property value declines to a constant value, denoted as H_*. As a result, the default, investment and financing decisions are then respectively satisfied the three equations given by:

$$H_*=\frac{\beta_2}{(\beta_2-1)}H_f<H_f,\quad \textit{where}\quad H_f=[M^*(1-\tau)-(1-e^{-rn})\frac{\delta\tau}{nr}]H^*, \tag{13}$$

$$-(1-\frac{1}{\beta_1})(M^* + (1-e^{-rn})\frac{\delta}{nr})H^*\tau - (\frac{1}{\beta_1}-\frac{1}{\beta_2})H_*(\frac{H^*}{H_*})^{\beta_2} + fM^{*\varepsilon} = 0\,, \tag{14}$$

and

$$\tau H^* + (1-\tau)H^*(\frac{H^*}{H_*})^{\beta_2} - f\varepsilon M^{*\varepsilon-1} = 0\,, \tag{15}$$

where it is required that $M^* > (1-e^{-rn})\delta\tau/(nr(1-\tau))$ to ensure that $H_* > 0$, and β_1 and β_2 are respectively given by

$$\beta_1 = \frac{1}{2} - \frac{r-\lambda}{\sigma^2} + \sqrt{(\frac{1}{2}-\frac{r-\lambda}{\sigma^2})^2 + \frac{2r}{\sigma^2}} > 1, \text{ and } \beta_2 = \frac{1}{2} - \frac{r-\lambda}{\sigma^2} - \sqrt{(\frac{1}{2}-\frac{r-\lambda}{\sigma^2})^2 + \frac{2r}{\sigma^2}} < 0\,. \tag{16}$$

Equation (13) is derived based on the condition that at the default trigger point the investment value is equal to zero, where the term H_f denotes the threshold value of the property that equates the loan balance. Equation (13) thus confirms the result of Kau et al. (1993), which suggests that given that an investor has an option value to default, the investor will not default even though the property value declines to this threshold. Equation (14) is derived based on the condition that an investor balances the immediate benefit from purchasing a property against the benefit from waiting for a more favorable state of nature. Equation (15) is derived based on the condition that an investor trades off the tax shield benefit against the sunk costs of debt financing. We can use Equations (13), (14) and (15) simultaneously to derive the solution for the choices of the LTV ratio, M^*, the investment trigger, H^*, and the default trigger, H_*.

As mentioned in the introduction section, the standard literature abstracts from the option to default. In our framework, we can compare the result that allows for it with the result that ignores it. In Equations (14) and (15) the values of the option to default are those terms associated with $(\frac{H^*}{H_*})^{\beta_2}$. Ignoring them yields

$$-(1-\frac{1}{\beta_1})(M^{*'} + (1-e^{-rn})\frac{\delta}{nr})H^{*'}\tau + fM^{*'\varepsilon} = 0, \tag{14'}$$

and

$$\tau H^{*'} - f\varepsilon M^{*'\varepsilon-1} = 0, \tag{15'}$$

where $H^{*'}$ denotes the investment trigger, and $M^{*'}$ denotes choices of the LTV ratio for this scenario. Solving Equations (14') and (15') simultaneously yields

$$M^{*'} = \frac{(1-e^{-rn})\frac{\delta\varepsilon}{nr}}{(1-\frac{1}{\beta_1})^{-1} - \varepsilon}, \tag{17}$$

and

$$H^{*'} = \frac{f\varepsilon}{\tau} M^{*'\varepsilon-1}, \tag{18}$$

where it is required that $0 < M^{*'} < 1$. In other words, it is required that $1-(1/\beta_1) < 1/\varepsilon$ and $(1-e^{-rn})\delta\varepsilon/(nr) < 1/(1-(1/\beta_1)) - \varepsilon$. The investment value net of the investment cost is then given by

$$NIV' = (\varepsilon - 1)M^{*'\varepsilon} + \frac{\tau\varepsilon}{nr}(1-e^{-rn})H^{*'}. \tag{19}$$

An investor certainly benefits from the option to default. However, in general we are not sure how this option affects the investor's incentives to borrow and invest, as demonstrated by Figure 1. In that figure, line T_nT_n denotes the relationship between $H^{*'}$ and $M^{*'}$ shown by Equation (14') and line D_nD_n denotes the relationship between $M^{*'}$ and $H^{*'}$ shown by Equation (15'). These two lines intersect at A_n, the equilibrium for the case where the investor doesn't have any option to default. Moreover, line TT denotes the relationship between H^* and M^* shown by Equation (14) and line DD denotes the relationship between H^* and M^* shown by Equation (15). The two lines intersect at A_0, the equilibrium for the case where the investor can default. The figure shows that line TT lies below line T_nT_n, thus indicating that, given the LTV ratio, an investor will purchase sooner when having the option to default. This, in turn, will induce the investor to borrow less because the investor purchases the property at a less favorable state of nature. In addition to the effects arising from the interaction of the default option and the investment decision stated above, we must also consider the effects arising from the interaction of the default option and the financing decision as stated below: Figure 1 shows that line DD lies to the right of line D_nD_n, thus indicating that, given the state of nature, an investor who owns the option to default will borrow more. This, in turn, will lead the investor to delay purchasing because waiting then becomes more valuable. Pooling together all information above yields indeterminate effects regarding how the existence of the option to default affects the investor's investment and financing decisions. Figure 1 depicts the case in which each of the two effects derived from the first interaction dominates its counterpart derived form the second.

Given that we allow the option to default, we are thus able to calculate the probability of default at any instant s after an investor purchases the property at the initial date. This probability is equal to the first passage time distribution for $H(0)$ to reach H_* before time s, and is thus given by (see, e.g., Leland, 2004; Kau et al., 2009)

$$L(H(0), H_*, s) = N(h_1(s)) + (\frac{H(0)}{H_*})N(h_2(s)), \tag{20}$$

where $N(\cdot)$ denotes cumulative normal distribution functions,

$$h_1(s) = \frac{-\ln\frac{H(0)}{H_*} - vs}{\sigma\sqrt{s}}, \quad h_2(s) = \frac{-\ln\frac{H(0)}{H_*} + vs}{\sigma\sqrt{s}}, \text{ and } v = r - \lambda - \frac{\sigma^2}{2}. \tag{21}$$

Substituting $H(0) = H^*$ into Equation (20), and letting s approach infinity yields

$$L(H^*, H_*, \infty) = 1, \text{ if } v \le 0, \tag{22}$$

$$= (\frac{H_*}{H^*})^{\frac{2v}{\sigma^2}} < 1, \text{ if } v > 0.$$

Equation (22) indicates that, the property value will eventually touch the default trigger point if the rate of return on housing investment is expected to be non-positive (i.e., $r - \lambda - \sigma^2 / 2 \le 0$). If, instead, the rate of return is expected to be positive, then the value of the property will sometimes, but not for sure touch the default trigger point over an infinite period of time.

Proposition 1 states how various exogenous forces affect the default trigger point characterized by Equation (13), and Proposition 2 states how these forces affect the default probability characterized by Equation (22), assuming that $r - \lambda - \sigma^2 / 2 > 0$.

Proposition 1. *An investor will default at a more favorable state of nature* (H_* *will increase*) *when* (*i*) *the investor borrows more* (M *increases*); (*ii*) *the investor is allowed to depreciate less capital per year* (n *increases*); (*iii*) *the property has less capital for deprecation* (δ *decreases*);

(*iv*) *the investor is taxed at a lower rate* (τ *decreases*); (*v*) *the property generates less service flow* (λ *decreases*); (*vi*) *housing price inflation becomes less volatile* (σ *decreases*); *and* (*vii*) *the mortgage rate increases* (r *increases*).

Proof: Differentiating H_* with respect to M, n, δ, τ, λ, σ, and r yields the results.

Proposition 2. *An investor is more likely to default* ($L(H^*, H_*, \infty)$ *will increase*) *when* (*i*) *the investor borrows more* (M *increases*); (*ii*) *the investor is allowed to depreciate less capital per year* (n *increases*); (*iii*) *the property has less capital for depreciation* (δ *decreases*); and (*iv*) *the investor is taxed at a lower rate* (τ *decreases*). *The likelihood for the investor to default is indeterminate, however, if* (*v*) *the property generates less service flow* (λ *decreases*); (*vi*) *housing price inflation becomes less volatile* (σ *decreases*); *and* (*vii*) *the mortgage rate increases* (r *increases*).

Proof: Differentiating $L(H^*, H_*, \infty)$ with respect to M, n, δ, τ, λ, σ, and r yields the results.

Equation (A5) in Appendix A indicates that the value from purchasing a property an investor receives, $F_2(H(t))$, consists of two parts, i.e., the value of the option to default (the sum of the first two terms), and the after-tax expected present value that abstracts from this option (the remaining terms). Propositions 1(i) – (iv) follow as a result of the decrease in the non-option value part: an increase in M will increase debt obligations, and an increase in n or a decrease in either δ or τ will reduce the tax shield benefit. Proposition 1(v) – (vi) follow as a result of the decrease in the value of the option to default when uncertainty in housing price inflation (σ) decreases or the service flow (λ) decreases (since which implies that the housing price is expected to infalte more, i.e., $r-\lambda$ will be higher). Finally, Proposition 1(vii) follows because an increase in the mortgage rate leads to both a lower non-option value resulting from a larger mortgage payment and a lower option value to default.

Proposition 2(i) – (iv) indicate that, it is more likely for an investor to default for those scenarios stated in Proposition 1(i) – (iv) because the default trigger point is closer to the investment trigger point. In additional to the above effect, for scenarios stated in Proposition 1(v) – (vii), we must also consider the following conflicting effect: It is less likely for the housing price to touch a given default trigger point, and thus less likely for an investor to default the loan when uncertainty is less significant, and when the housing price is expected to inflate more, either because the service flow rate decreases or the mortgage rate increases.[11] That is why we derive indeterminate results in Proposition 2(v) – (vii).[12]

We can compare our results of Propositions 1 and 2 with those of Kau et al. (2009), who build a more simplified model than ours because they abstract from taxation. Although the mortgage literature (see, e.g., Quercia and Stegman, 1992) consistently shows that the initial LTV is the main factor affecting the default decision, yet they argue that the current LTV ratio rather than the initial LTV ratio matters. They show that a loan borrower's default decision can be described by a "barrier" of the current LTV ratio such that the borrower will default once this barrier, denoted by LTV^*, is touched. We may also derive this barrier by dividing the default trigger, H_*, by the initial loan amount, H^*M^*, thus yielding

$$LTV^* = \frac{\beta_2}{(\beta_2 - 1)}[(1-\tau) - (1-e^{-nr})\frac{\delta\tau}{nrM^*}]. \tag{13'}$$

Differentiating LTV^* with respect to its underlying parameters yields similar conclusion as stated in Proposition 1 except for the effect of the initial LTV ratio, M^*. We find that LTV* is negatively related to the initial LTV ratio, which results from the fact that as the initial LTV ratio increases (M^* increases), the bankruptcy trigger (H_*) also increases (see Proposition 1(i)), but less than proportionally, such that LTV* (H_* / H^*M^*) decreases. In other words, we still find that an increase in the initial LTV raises the default barrier.

However, in the absence of any taxation ($\tau = 0$), we find that LTV^* will not be related to the initial LTV, thus confirming the result of Kau et al. (2009), who argue that one can predict the default probability by ignoring the initial LTV, and instead, by classifying the type of property whose value may evolve differently. Our result of Proposition 2 supports their latter assertion because we can classify the property type by the expected capital gain ($r - \lambda$) from holding it

[11] Our paper uses a structural-form model, thus contrasting with Kau, Keenan and Smurov (2011), which uses a reduced form model to investigate the relationship between the LTV ratio and the probability of default.

[12] Proposition 2(i), which states that the mortgage default probability is increasing with the LTV ratio, is supported by Green et al. (2010). Furthermore, similar to Foster and Van Order (1984), we find that the option to default is more valuable as housing price inflation becomes more volatile, as suggested by Proposition 1(vi). However, Proposition 2(vi) indicates that this does not necessarily imply that the default probability is increasing with the volatility of property value. In other words, our result of Proposition 2(vi) raises doubt about using the volatility of property value to proxy the default probability, as is usually implemented in the empirical study (see, e.g., the discussion in Green et al., 2010).

and the volatility of that capital gain (σ). Nevertheless, our result also suggests that the initial LTV ratio and factors related to tax shield benefit also matter in a world with taxation.

While Propositions 1 and 2 offer several testable implications regarding the determinants of the default decision, yet we find that only one exogenous force, namely, the sunk cost, will exhibit unambiguous impacts on choices of the investment timing, and the LTV ratio as stated below.

Proposition 3. *An investor who incurs larger sunk costs when applying for a mortgage loan to purchase a property will purchase later and default earlier, but will leave unchanged the LTV ratio and the likelihood to default.*

Proof: See Appendix B.

The intuition behind Proposition 3 is explained by using Figure 2. Let us start from an initial equilibrium point A_0, the intersection of lines TT and DD, where line TT depicts the relationship between H^* and M^* addressed in Equation (14), and line DD depicts the relationship between M^* and H^* addressed in Equation (15). As a result, an investor will purchase a property once the property value reaches H_0^*, and at that instant, will borrow at a LTV ratio equal to M_0^*. Now consider that an investor incurs a larger sunk cost from applying for a mortgage loan. Given the LTV ratio, the investor will gain less from the purchase, and will thus wait longer. This is shown by a shift of line TT upward to line $T'T'$. On the other hand, given the state of nature, the investor will borrow less because the investor will suffer more from a higher marginal cost of debt financing. This is shown by a shift of line DD leftward to line $D'D'$. The new equilibrium point, A_1, the intersection of lines $T'T'$ and $D'D'$, indicates that the investor will purchase later, but will not alter the LTV ratio. Furthermore, since the investor delays the purchase, the default trigger point, H_*, will also move upward (because H^* is positively related to H_* as indicated in Equation (13)), but the probability of default will remain unchanged as the wedge between H^* and H_* remains unchanged.

The comparative-statics results of the other exogenous forces on the investment and financing decisions are all indeterminate, and thus in the next section we employ plausible parameters to numerically clarify them. We will consider both cases, that is, where the holding period is infinite and where it is finite. Appendix C shows the procedures to find the solution for the latter case.

III. Numerical Analysis

The benchmark case we choose is as follows: The parameter for the cost function $f = 1$ unit, the income tax rate $\tau = 20\%$; the number of years allowed for depreciation $n = 39$ years; the proportion of depreciable capital $\delta = 50\%$; the mortgage rate $r = 7.5\%$ per year; the service flow rate $\lambda = 5\%$ per year; the cost elasticity of debt financing $\varepsilon = 1.5$; the volatility of housing price inflation $\sigma = 12.5\%$ per year; and the holding period $\bar{t} = \infty$.[13]

Insert Figures 3 and 4 here

To fully explain how the option to default affects the investment timing, default timing, debt

[13] According to Goetzmann and Ibbotson (1990), during the period of 1969 to 1989, the annual standard deviation for REITs on commercial property was equal to 15.4%. The volatility of housing price inflation in our benchmark case is a little smaller than this value.

financing decisions, and investment performance, Figure 3 changes σ in the region (8%, 22.4%), holding all the other parameters at their benchmark values.[14] The figure indicates that, given the benchmark parameter values (i.e., $\sigma = 12.5\%$), an investor will not purchase a property until the property value reaches 5.662 units ($H^* = 5.662$). At that instant, the investor will choose a LTV ratio equal to 65.5% ($M^* = 65.5\%$), and will gain 0.404 units (*NIV* = 0.404).[15] Thereafter, the investor will not default until the property value declines to 2.267 units ($H_* = 2.267$). Thus, at the instant at which an investor purchases a property, the investor expects to have 13.4% chance to eventually default the loan ($L = 13.4\%$).

Figure 3 also indicates that, without any option to default an investor will not purchase a property until the property value exceeds H^* by 11.6%, i.e., 6.32 units ($H^{*'} = 6.32$). At that instant, the investor will apply for a LTV ratio that exceeds M^* by 6.8%, i.e., 71.0% ($M^{*'} = 71.0\%$), and will gain at an amount that is short of *NIV* by 2.0%, i.e., 0.396 units ($NIV' = 0.396$).[16] These results fit into Figure 1, which shows that the existence of the option to default encourages an investor not only to purchase earlier, but also to borrow less.

Figure 3 also shows that an investor who expects housing price inflation to be more volatile (σ increases) is more likely to default (L increases), and will purchase earlier (H^* decreases), default later (H_* decreases), borrow less (M^* decreases) and gain more (*NIV* increases). We use

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[14] We consider the permissible region for σ: When $\sigma = 8\%$ per year, the LTV ratio is equal to 100%, while when $\sigma = 22.4\%$ per year, the probability for an investor to eventually default the loan starting from the date of purchasing is equal to 100%.

[15] The transaction cost as a proportion of the property value, $fM^{*\varepsilon} / H^*$, is equal to 9.36%. This is near the ratio, i.e., 10% estimated by Stokey (2009, p.188), which includes the commissions for real estate agents (5-6%) and other costs such as time spent on searching, moving costs, and the like (4-5%).

[16] In this section, we evaluate the net value of investment at the date of purchasing for the benchmark case, i.e., 5.662. In other words, we use the factor $(5.662 / H^*)^{\beta_1}$ to adjust for any scenario that yields a value of H^* differing from 5.662 such that $NIV' = NIV \cdot (5.662/H^*)^{\beta_1}$. This factor is also equal to the probability for an investor who purchases a property later than the benchmark case.

Figure 4 to explain these results. Suppose that point A_0 denotes the initial equilibrium, the intersection of lines TT and DD. Greater uncertainty in housing price inflation confers an investor more option value to default, thus accelerating investment by shifting line TT downward to line $T'T'$. Greater uncertainty also encourages (discourages) borrowing when the LTV ratio exceeds (shorts of) a critical level, denoted by A_2. This is shown by line DD that shifts to line $D'D'$ by rotating clockwise around point A_2. The new equilibrium, point A_1, the intersection of lines $T'T'$ and $D'D'$, indicates that the investor will purchase sooner and borrow less, as compared to the initial equilibrium, point A_0. As suggested by Proposition 1(i), The default trigger point also shifts downward mainly because the investor borrows less. Greater uncertainty also raises the likelihood to default as the default trigger point is more likely to be hit. The net investment value also increases with uncertainty, given that the investor has more leeway to default.

Insert Table 1 here

Table 1 shows the results for f changing in the region (0.5, 1.5), τ in the region (15%, 25%), n in the region (37, 41), δ in the region (0.4, 0.6), r in the region (6.5%, 8.5%), λ in the region (4%, 6%), ε in the region (1.4, 1.6), and $\bar{t}$ in the region of $(5,\ 25)$, holding all the other parameters at their benchmark values. Table 1 indicates the following results. First, an investor will wait longer to purchase a property (H^* increases), but will alter neither the LTV ratio nor the likelihood to default (both M^* and $L(H^*, H_*, \infty)$ remain unchanged) if the transaction cost (f) increases. This conforms to the conclusion in Proposition 3. As expected, the default trigger (H_*) increases with the investment trigger H^*, and the investor also gains less when the sunk cost increases.

Insert Figures 5 - 7 here

Second, an investor who faces a higher tax rate (τ increases) is less likely to default (L decreases), and will invest earlier (H^* decreases), default later (H_* decreases), and borrow more (M^* increases), thus gaining more from the purchase. We use Figure 5 to explain these results. Suppose that point A_0 denotes the initial equilibrium. An investor who faces a higher tax rate will borrow more because the investor can enjoy more tax shield benefits. This is shown by a shift from line DD rightward to line $D'D'$. On the other hand, the investor will also purchase earlier because waiting will then be less valuable. This is shown by a shift from line TT downward to line $T'T'$. The new equilibrium is at point A_1, which indicates that the investor will borrow more and purchase earlier, as compared to the initial equilibrium, point A_0. Both the default trigger point and the default probability decrease mainly because the investor purchases at a better state of nature. The investor also gains more as a result of injecting less money out of pocket.

Third, an investor is less likely to default (L decreases), will invest earlier (H^* decreases), default later (H_* decreases), borrow less (M^* decreases), and thus gain less from the purchase, if (i) the investor is allowed to depreciate less rapidly (n increases); (ii) the proportion of depreciable capital decreases (δ decreases); and (iii) the sunk cost responds less to a change in the debt level (ε decreases). We use Figure 6 to explain these results. Proposition 1 indicates that under these scenarios, the option value to default will decrease such that an investor will borrow less, given the state of nature. This is captured by a leftward shift from line DD to line $D'D'$. On the other hand, waiting will become less valuable, which is captured by a downward shift form line TT to $T'T'$. As indicated by the new equilibrium, point A_1, the investor will accelerate the purchase, but will borrow less, as compared to the initial equilibrium, point A_0. Given that the investor borrows less, the default trigger will decrease and the investor is also less likely to default. The investor will also gain less as having less option value to default.

Fourth, an increase in the mortgage rate (r) or a decrease in the service flow rate (λ) will induce an investor to be less likely to default (L decreases), borrow less (M^* decreases), default later (H_* decreases), and gain less, but will exhibit an indeterminate effect on the investment timing. Given that the mechanism for deriving these results is quite complicated, we will thus only explain a special case, namely, the service flow rate is decreased from 5% to 4% per year. Let us start from the initial equilibrium, point A_0, the intersection of lines TT and DD. As λ decreases, the option value to default decreases, as indicated by Proposition 1. Consequently, the investor will have more option value from waiting, as indicated by an upward shift from line TT to $T'T'$, but will benefit less from borrowing, as indicated by a shift from line DD leftward to $D'D'$. The new equilibrium, point A_1, indicates that the investor will purchase sooner and borrow less as a result. Given that the investor borrows less, the investor will thus default later as well as be less likely to default, and gain less as injecting more money out of pocket.

Finally, let us consider the case where an investor borrows a finitely-lived mortgage loan. When the loan lasts longer, the investor will have more valuable default option, thus resembling the effects as those shown in Figure 4. In other words, an investor who applies for a loan that lasts longer will borrow less, purchase earlier, and thus gain more from the purchase. We thus predict that, an investor will prefer a long-lived to short-lived fixed-rate mortgage loan if the investor has the option to choose between them.

IV. Conclusion

This article has investigated the determinants of optimal capital structure in real estate investment in a real options framework where an investor incurs transaction costs when purchasing a property through borrowing. The literature (see, e.g., Follain, 1990) focuses on how characteristics of an investor such as length of stay, age, and income affect the LTV ratio chosen by the investor. By contrast, we focus on how characteristics related to demand and supply conditions of real estate markets affect the LTV ratio chosen by an investor. In particular, we first predict that the chosen LTV ratio will increase with the tax rate and the portion of capital allowed for depreciation, both of which accord well with the findings of Gau and Wang (1990). Second, we find that an investor will choose a lower LTV ratio if the investor’s option value to default becomes more valuable either because housing price inflation become more volatile or because the investor applies for a loan that lasts longer.

This article employs a simplified model, and thus can be extended in the following respects. First, we may assume that future interest rates are also stochastic over time, thus allowing for the prepayment option as addressed in Hilliard et al. (1998) and Schwartz and Torous (1992). Second, we can endogenize the discount point as addressed in Cannaday and Young (1995).

Finally, we may allow the mortgage rate to be adjustable as addressed in Kau et al. (1993), thus allowing an investor to choose between a fixed-rate and an adjustable-rate mortgage loan (see, e.g., Follain, 1990).

Appendix A: The case for $T = \infty$

When $T = \infty$, then $\partial V^e(\cdot)/\partial t = 0$, and $E_t ATER(T) e^{-r(T-t)} = 0$. Suppose that $F_2(H(t))$ denotes the value of $V(H(t),t)$ in the region where $t \geq 0$ and $H(t) > H_*$, where H_* is the critical level of the property value that triggers default at $t = 0$. As a result, we can rewrite Equation (6) as:

$$\frac{\sigma^2}{2} H(t)^2 \frac{\partial^2 F_2(H(t))}{\partial H(t)^2} + (r-\lambda)H(t)\frac{\partial F_2(H(t))}{\partial H(t)} + \lambda H(t) + \frac{\tau\delta}{n} H(0) - (1-\tau)rMH(0) \quad \text{(A1)}$$

$$= rF_2(H(t)).$$

Substituting $F_2(H(t)) = H(t)^\beta$ into the homogenous part of Equation (A1) yields the quadratic equation for solving β:

$$\phi(\beta) = -\frac{\sigma^2}{2}\beta(\beta-1) - (r-\lambda)\beta + r = 0. \quad \text{(A2)}$$

The solution to the homogenous part of Equation (A1) is thus given by

$$F_2(H(t)) = B_1 H(t)^{\beta_1} + B_2 H(t)^{\beta_2}, \quad \text{(A3)}$$

where β_1 and β_2 are, respectively, the larger and smaller roots of β in Equation (A2), and B_1 and B_2 are constants to be determined. The solution to the non-homogeneous part of Equation (A1) is given by

$$F_2(H(t)) = H(t) - (1-\tau)MH(0) + \frac{\tau\delta}{nr}(1-e^{-rn})H(0). \quad \text{(A4)}$$

The solution for $F_2(H(t))$ in Equation (A2) is thus given by:

$$F_2(H(t)) = B_1 H(t)^{\beta_1} + B_2 H(t)^{\beta_2} + H(t) - (1-\tau)MH(0) + \frac{\tau\delta}{nr}(1-e^{-rn})H(0). \quad \text{(A5)}$$

The terms B_1, B_2, and H_*, are solved simultaneously from the boundary conditions as follows:

$$\lim_{H(t)\to\infty} B_1 H(t)^{\beta_1} + B_2 H(t)^{\beta_2} = 0, \quad \text{(A6)}$$

$$F_2(H_*) = 0, \quad \text{(A7)}$$

and

$$\frac{\partial F_2(H_*)}{\partial H(t)} = 0. \quad \text{(A8)}$$

Equation (A6) is the limit condition, which states that the investor's option value to default is worthless as the housing price approaches infinity. This condition requires that $B_1 = 0$. Equation (A7) is the value-matching condition, which states that, at the optimal timing of default, the investor has a value equal to zero, as suggested by Leland (1994). Equation (A8) is the smooth-pasting condition, which guarantees that the investor will not derive any arbitrage profits by deviating the optimal default strategy. Solving equations (A6)-(A8) simultaneously yields

$$H_* = \frac{[M(1-\tau)-(1-e^{-rn})\frac{\tau\delta}{nr}]H(0)\beta_2}{(\beta_2 - 1)}, \tag{A9}$$

and

$$B_2 = -\frac{1}{\beta_2} H_*^{1-\beta_2}. \tag{A10}$$

Denote $F_1(H(t))$ as the option value of waiting at the region where investment has not been made ($H(t) < H^*$), which satisfies Equation (10), i.e.,

$$\frac{\sigma^2}{2} H(t)^2 \frac{d^2 F_1(H(t))}{dH(t)^2} + (r-\lambda)H(t)\frac{dF_1(H(t))}{dH(t)} = rF_1(H(t)). \tag{A11}$$

The solution for $F_1(H(t))$ is given by

$$F_1(H(t)) = A_1 H(t)^{\beta_1} + A_2 H(t)^{\beta_2}. \tag{A12}$$

The terms A_1, A_2, and H^* are solved from the boundary conditions as follows:

$$\lim_{H(t)\to 0} F_1(H(t)) = 0, \tag{A13}$$

$$F_1(H^*) = F_2(H^*) - C(H^*, M^*), \tag{A14}$$

and

$$\frac{\partial F_1(H^*)}{\partial H(0)} = \frac{\partial F_2(H^*)}{\partial H(0)} - \frac{\partial C(H^*, M^*)}{\partial H(0)}, \tag{A15}$$

where M^* is the choice of the LTV ratio. Equation (A13) is the limit condition, which indicates that the option from waiting is worthless if the housing price approaches its minimum permissible of zero. This requires that $A_2 = 0$. Equation (A14) is the value-matching condition, which states that at the optimal timing of investment, the investor is indifferent between purchasing and not purchasing. Equation (A15) is the smooth-pasting condition, which prevents the investor from deriving any arbitrage profits by deviating the optimal investment strategy.

Solving Equations (A13)-(A15) simultaneously yields

$$X(H^*, M^*) = -(1-\frac{1}{\beta_1})[M^* + \frac{\delta}{nr}(1-e^{-rn})]H^*\tau - (\frac{1}{\beta_1} - \frac{1}{\beta_2})(\frac{H^*}{H_*})^{\beta_2} H_* + fM^{*\varepsilon} = 0 . \qquad \text{(A16)}$$

For H^* to be an interior solution, it is required that the second-order condition holds, i.e., the derivative of the left-hand side of Equation (A16) with respect to H^* must be negative:

$$\Delta_{11} = \partial X(H^*, M^*) / \partial H^* < 0 . \qquad \text{(A17)}$$

Furthermore, the choice of M is found by setting the derivative of $F_1(H^*)$ in Equation (A14), or equivalently, $F_2(H^*) - C(H^*, M^*)$, with respect M equal to zero. This yields

$$Y(H^*, M^*) = \tau H^* + (1-\tau)H^*(\frac{H^*}{H_*})^{\beta_2} - f\varepsilon M^{*\varepsilon-1} = 0 . \qquad \text{(A18)}$$

The following second-order condition is also required to hold for M^* to be an interior solution:

$$\Delta_{22} = \partial Y(H^*, M^*) / \partial M^* < 0 . \qquad \text{(A19)}$$

QED

Appendix B: Proof of Proposition 2

Defining the left-hand side of Equation (14) as $X(H^*, M^*)$ and that of Equation (15) as $Y(H^*, M^*)$, and then totally differentiating these two equations with respect to f yields

$$\Delta_{11}\frac{\partial H^*}{\partial f} + \Delta_{12}\frac{\partial M^*}{\partial f} + \Delta_{13} = 0 , \qquad \text{(B1)}$$

$$\Delta_{21}\frac{\partial H^*}{\partial f} + \Delta_{22}\frac{\partial M^*}{\partial f} + \Delta_{23} = 0 , \qquad \text{(B2)}$$

where $\Delta_{11} = \partial X / \partial H^*$, $\Delta_{12} = \partial X / \partial M^*$, $\Delta_{13} = \partial X / \partial f$, $\Delta_{21} = \partial Y / \partial H^*$, $\Delta_{22} = \partial Y / \partial M^*$, and $\Delta_{23} = \partial Y / \partial f$. As a result,

$$\frac{\partial H^*}{\partial f} = (-\Delta_{13}\Delta_{22} + \Delta_{12}\Delta_{23}) / \Delta > 0 , \qquad \text{(B3)}$$

$$\frac{\partial M^*}{\partial f} = (-\Delta_{11}\Delta_{23} + \Delta_{21}\Delta_{13}) / \Delta = 0 , \tag{B4}$$

where $\Delta = \Delta_{11}\Delta_{22} - \Delta_{21}\Delta_{21} > 0$. For H^* and M^* to be interior solutions, it is required that $\Delta > 0$, which also indicates that the slope of DD must be steeper than that of TT .

QED

Appendix C: The case for finite $\bar{t}$

We follow Brennan and Schwartz (1978) and Hull and White (1990) to find H^* and M^*. Let $y = \ln H$ such that $y^* = \ln H^*$, $U(y) = F_1(H)$, and $Z(y,t) = V(H,t)$. As a result, Equation (10) can be rewritten as:

$$\frac{\sigma^2}{2}\frac{dU^2(y)}{dy^2} + (r - \lambda - \frac{\sigma^2}{2})\frac{dU(y)}{dy} - rU(y) = 0 \text{, if } y(0) < y^* . \tag{C1}$$

Similarly, Equation (5) can be rewritten as:

$$\frac{\sigma^2}{2}\frac{\partial^2 Z(y,t)}{\partial y^2} + (r - \lambda - \frac{\sigma^2}{2})\frac{\partial Z(y,t)}{\partial y} + \frac{\partial Z(y,t)}{\partial t} + \lambda e^y + [-(1-\tau)rM + \frac{\delta\tau}{n}]e^{y^*} = rZ(y,t) , \tag{C2}$$

if $t \geq 0$, and $y(0) \geq y^*$.

Furthermore, Equation (6) can be rewritten as:

$$Z(y(T),T) = e^{y(T)}(1-\tau) - e^{y^*}(M + \frac{\tau\delta T}{n} - \tau) . \tag{C3}$$

Finally, Equation (2) can be rewritten as:

$$C(e^{y^*}, M) = (1-M)e^{y^*} + fM^{\varepsilon} . \tag{C4}$$

Let

$$G(H,t) = \partial V(H,t) / \partial M . \tag{C5}$$

Differentiating Equation (5) term by term with respect to M yields

$$\frac{\sigma^2}{2}H^2\frac{\partial^2 G(H,t)}{\partial H^2} + (r-\lambda)H\frac{\partial G(H,t)}{\partial H} + \frac{\partial G(H,t)}{\partial t} - (1-\tau)rH^* = rG(H,t) . \tag{C6}$$

The choice of M , denoted by M^*, is derived by setting the derivative of $W = V(\cdot) - C(\cdot)$ with respect to M equal to zero. This yields

$$\frac{\partial W}{\partial M} = G(H^*, 0) + H^* - f\varepsilon M^{*\varepsilon-1} = 0 . \tag{C7}$$

Let $g(y,t) = G(H,t)$. As a result, Equation (C7) can be rewritten as

$$g(y^*, 0) + e^{y^*} - f\varepsilon M^{*\varepsilon-1} = 0 . \tag{C8}$$

Furthermore, Equation (C6) can be transformed into:

$$\frac{\sigma^2}{2}\frac{\partial^2 g(y,t)}{\partial y^2} + (r - \lambda - \frac{\sigma^2}{2})\frac{\partial g(y,t)}{\partial y} + \frac{\partial g(y,t)}{\partial t} - rg(y,t) - (1-\tau)re^{y^*} = 0 . \tag{C9}$$

We can implement the explicit finite difference method (Hull and White, 1990) to solve for M^* and H^*. We begin by choosing a small time interval, Δt, and a small change in y, Δy. A grid is then constructed for considering values of g when y is equal to

y_0, $y_0 + \Delta y$, …, $y_{\max}$,

and time is equal to

0, Δt, …, T .

The parameters y_0 and $y_{\max}$ are the smallest and the largest values of y, respectively, and $t = 0$ is the current time. We denote $y_0 + i\Delta y$ by y_i ($i = 1, \cdots, n$), $j\Delta t$ by t_j ($j = 1, \cdots, m$), and the value of g at the (i, j) point on the grid by $g_{i,j}$. The partial derivatives of g with respect to y at node (i, j) are approximately as follows:

$$\frac{\partial g}{\partial y} = \frac{g_{i+1,j} - g_{i-1,j}}{2\Delta y}, \tag{C10}$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{g_{i+1,j} + g_{i-1,j} - 2g_{i,j}}{(\Delta y)^2}, \tag{C11}$$

and the time derivative is approximately

$$\frac{\partial g}{\partial t} = \frac{g_{i,j} - g_{i,j-1}}{\Delta t} . \tag{C12}$$

Equation (C8) indicates that $e^{y^*} = [f\varepsilon M^{*\varepsilon-1} - g(y^*, 0)]$. Substituting this and Equations (C10) – (C12) into Equation (C9) yields:

$$g_{i,j-1} = ag_{i-1,j} + bg_{i,j} + cg_{i+1,j} - \frac{\Delta t}{(1 + r\Delta t)} r(1-\tau)[f\varepsilon M^{*\varepsilon-1} - g_{i^*,0}] , \tag{C13}$$

where

$$a=\frac{1}{(1+r\Delta t)}[\frac{\sigma^2\Delta t}{2(\Delta y)^2}-(r-\lambda-\frac{\sigma^2}{2})\frac{\Delta t}{2\Delta y}], \tag{C14}$$

$$b=\frac{1}{(1+r\Delta t)}[1-\frac{\sigma^2\Delta t}{(\Delta y)^2}], \tag{C15}$$

$$c=\frac{1}{(1+r\Delta t)}[\frac{\sigma^2\Delta t}{2(\Delta y)^2}+(r-\lambda-\frac{\sigma^2}{2})\frac{\Delta t}{2\Delta y}], \tag{C16}$$

and i^* is the value of i for $g_{i,0}$ to reach the lowest level. The choice of M, M^*, is the one that makes the left-hand side of Equation (C13) equal to $g_{i^*,0}$.

Similarly, Equation (C2) can be rewritten as:

$$Z_{i,j-1}=aZ_{i-1,j}+bZ_{i,j}+cZ_{i+1,j}+\frac{\Delta t}{(1+r\Delta t)}\lambda e^{y_i}+\frac{\Delta t}{(1+r\Delta t)}[rM(-1+\tau)+\frac{\delta\tau}{n}]e^{y^*}, \tag{C17}$$

where $Z_{i^*,0}$ is found at $i=i^*$ such that the left-hand side of Equation (C17), $Z_{i,j-1}$, is equal to $Z_{i^*,0}$. For any i, the limited liability of equity requires that

$$Z_{i,j-1}\geq 0. \tag{C18}$$

Furthermore, at the maturity date T, Equation (C3) indicates that $Z_{i,m}$ must satisfy

$$Z_{i,m}=e^{y_i}(1-\tau)-e^{y_{i^*}}[M^*+\frac{\tau\delta T}{n}-\tau]. \tag{C19}$$

We also need to consider the condition for choices of the investment timing. The solution to $U(y)$ in Equation (C1) is given by

$$U(y)=A_1e^{\beta_1 y}+A_2e^{\beta_2 y}, \tag{C20}$$

where A_1 and A_2 are constants to be determined, and β_1 and β_2 are defined in Appendix A. The optimal investment timing is determined by the following boundary conditions:

$$\lim_{y\to 0}U(y)=0, \tag{C21}$$

$$U(y^*)=Z(y^*,0)-(1-M)e^{y^*}-fM^{*\varepsilon}, \tag{C22}$$

and

$$\frac{dU(y^*)}{dy} = \frac{\partial Z(y^*,0)}{\partial y} - (1-M)e^{y^*}. \tag{C23}$$

Solving Equations (C21)-(C23) simultaneously yields

$$A_2 = 0, \tag{C24}$$

$$A_1 = (Z^e(y^*,0) - (1-M)e^{y^*} - fM^{*\varepsilon}) / e^{\beta_1 y^*}, \tag{C25}$$

and

$$A_1\beta_1 e^{\beta_1 y^*} = \frac{\partial Z(y^*,0)}{\partial y} - (1-M)e^{y^*}. \tag{C26}$$

The law of motion for $Z(y,t)$ shown in Equation (C2) and that for $g(y,t)$ shown in Equation (C9) are subject to two optimal conditions shown in Equations (C8) and (C26), respectively, the constraint shown in Equation (C18) and the boundary condition shown (C19). Solving these conditions simultaneously yields the solutions for M^*, $g_{i^*,0}$, and $Z_{i^*,0}$, where $Z_{i^*,0}$ is the investment value in equilibrium and $H^* = e^{y_{i^*}}$.

QED

Table 1

Optimal Investment and Financial Policies for Different Economic and Financial Variables

This table reports the levels of the net investment value ($V-(1-M^*)H^*-fM^{*\varepsilon}$), as well as the default probability ($L(H^*, H_*, \infty)$), when the investment trigger (H^*), the LTV ratio (M^*), and the default trigger (H_*) are all chosen to maximize the net value of investment. Panel A reports the results for the base case. Panel B reports the results when the sunk cost of investment (f) is equal to 0.5 and 1.5. Panel C reports the results when the corporate tax rate (τ) is 15% and 25%. Panel D reports the results when years allowed for depreciation are equal to 37 and 41. Panel E reports the results when the depreciable capital (δ) accounts for 40% and 60%. Panel F reports the results when the mortgage rate (r) is equal to 6.5% and 8.5% per year. Panel G reports the results when the service flow rate (λ) is equal to 4% and 6% per year. Panel H reports the results when the cost elasticity of debt financing (ε) is equal to 1.4 and 1.6. Panel I reports the results when the maturity is 5, 10, 15, and 25 years. All Panels are reported by holding all other parameters at their benchmark values. The net investment has adjusted as stated in footnote 15.

	Loan-to-Value Ratio, M^* (%)	Investment trigger, H^*	Default trigger, H_*	Net investment value, $V-(1-M^*)H^*-fM^{*\varepsilon}$	Default Probability, $L(H^*, H_*, \infty)$ (%)
A. Benchmark Case					
f =1, τ =20%, δ =50%, r =7.5% λ =5%, ε =1.5, σ =12.5%	65.5	5.662	2.267	0.404	13.4
B. Sunk Cost of Investment:					
f =0.5	65.5	2.831	1.134	0.902	13.4
f =1.5	65.5	8.493	3.401	0.404	13.4

Table 1 (continued)

Optimal Operating and Financial Policies for Different Economic and Financial Variables

	LTV Ratio, M^* (%)	Investment trigger, H^*	Default trigger, H_*	Net investment value, $V-(1-M^*)H^*-fM^{*\varepsilon}$	Default Probability, $L(H^*,H_*,\infty)$ (%)
C. Corporate Tax Rate:					
τ =15%	62.8	7.079	2.939	0.218	14.5
τ =25%	67.4	4.726	1.791	0.653	11.8
D. Years Allowed for Depreciation:					
n =37	67.6	5.697	2.354	0.412	14.3
n =41	63.4	5.620	2.181	0.396	12.5
E. Depreciable Capital:					
δ =40%	54.7	5.37	1.801	0.365	9.0
δ =60%	74.4	5.755	2.607	0.441	17.5
F. Contract Rate of Interest:					
r =6.5%	84.0	5.055	2.479	0.435	51.9
r =8.5%	42.6	4.881	1.286	0.376	1.0

Table 1 (continued)

Optimal Operating and Financial Policies for Different Economic and Financial Variables

	LTV Ratio, M^* (%)	Investment trigger, H^*	Default trigger, H_*	Net investment value, $V-(1-M^*)H^*-fM^{*\varepsilon}$	Default Probability, $L(H^*, H_*, \infty)$ (%)
G. Service Flow Rate:					
λ =4%	33.3	4.323	0.850	0.384	0.4
λ =6%	88.7	5.016	2.657	0.404	55.7
H. Cost Elasticity of Debt Financing:					
ε =1.4	50.1	5.202	1.560	0.375	7.1
ε =1.6	84.4	5.856	3.067	0.424	24.1
I. Maturity:					
T = 5	77.9	6.178		0.037	
T =10	77.5	6.073		0.060	
T =15	76.1	5.917		0.123	
T =25	72.0	5.708		0.247	

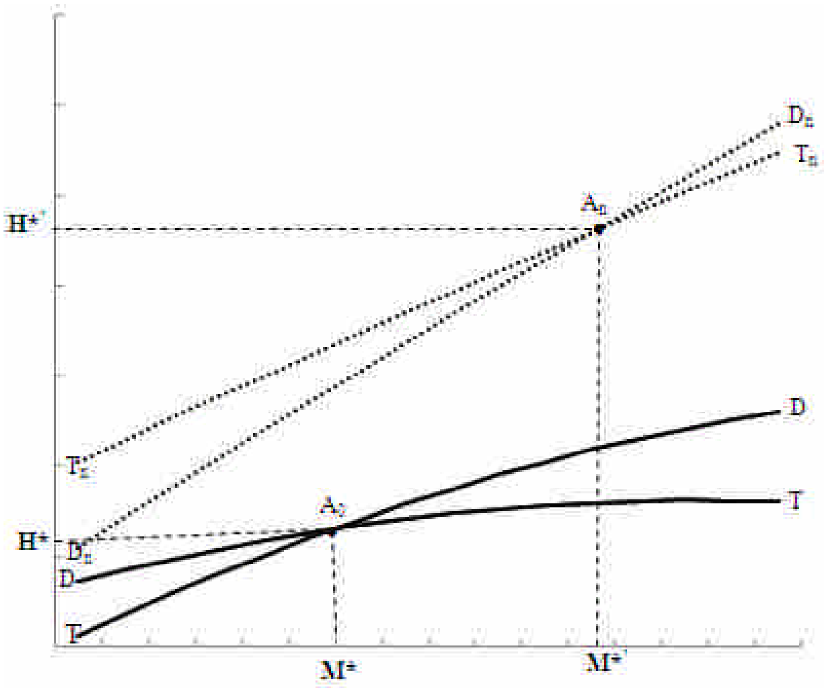

Figure 1. The impacts of the existence of the option to default on the investment and financing decisions. The graph shows that the existence of the option to default will move the lines that characterize the investment decision and financing decisions from T_nT_n to TT, and from D_nD_n to DD, respectively. The equilibrium will thus move from point A_n to point A_0, indicating that an investor who has the option to default will purchase sooner and borrow less.

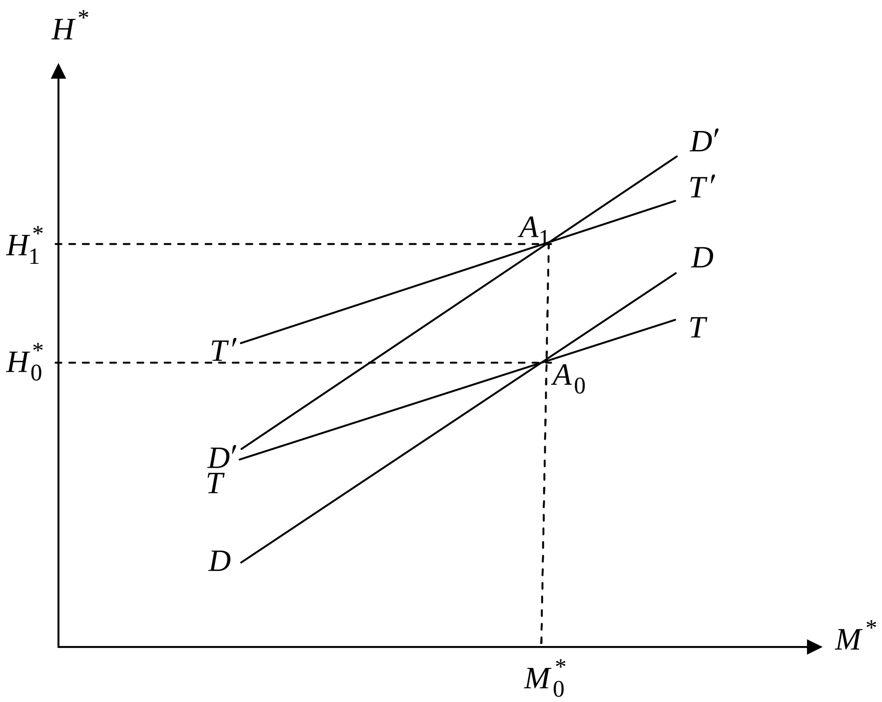

Figure 2. An increase in the sunk cost of investment, f, will delay investment, but will leave choices of the LTV ratio unchanged. The graph shows that the initial equilibrium, point A_0, is the intersection of line TT (which characterizes the optimal condition for choices of the investment timing) and line DD (which characterizes the optimal condition for choices of the loan-to-value ratio). An increase in f will shift line TT upward to line $T'T'$ and line DD leftward to line $D'D'$. The new equilibrium is at point A_1, which indicates that the critical level of the housing price that triggers investment will move upward to H_1^*, but choices of the LTV ratio will remain unchanged at M_0^*.

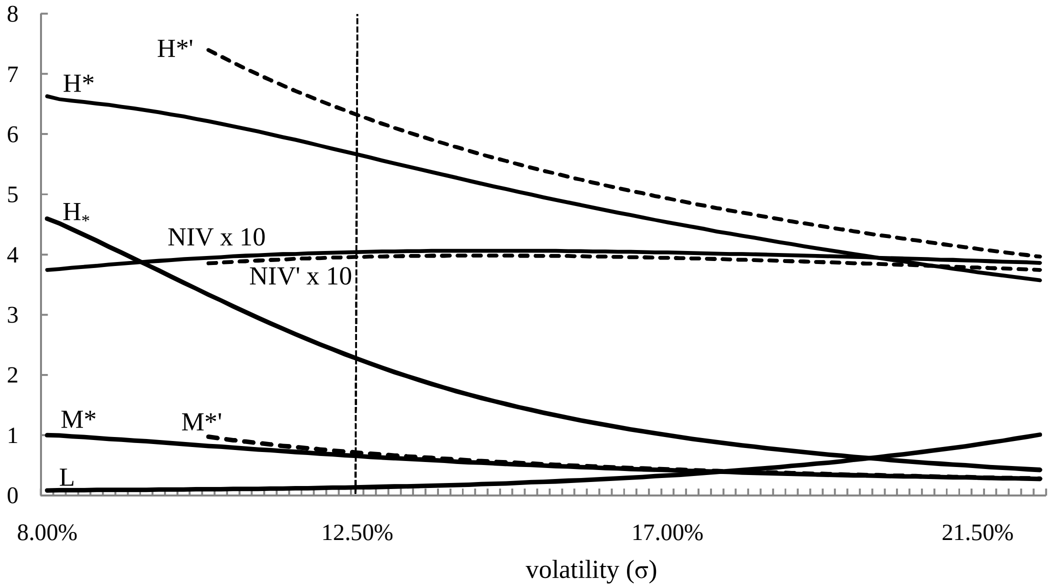

Figure 3. Comparison of the case with the option to default and that without it for different volatilities of housing price inflation, σ. In this graph, the default trigger is denoted by H_*, and the likelihood to default is denoted by L. The LTV ratios with and without the default option are denoted by M^* and $M^{*'}$, respectively. The investment triggers with and without the default option are denoted by H^* and $H^{*'}$, respectively. The net investment values with and without the default option are denoted by NIV and NIV', respectively, where we have discounted these two values at the point of investment for the benchmark case ($\sigma = 12.5\%$) in which an investor has the option to default. The graph shows that $M^{*'} > M^*$, $H^{*'} > H^*$, and $NIV > NIV'$.

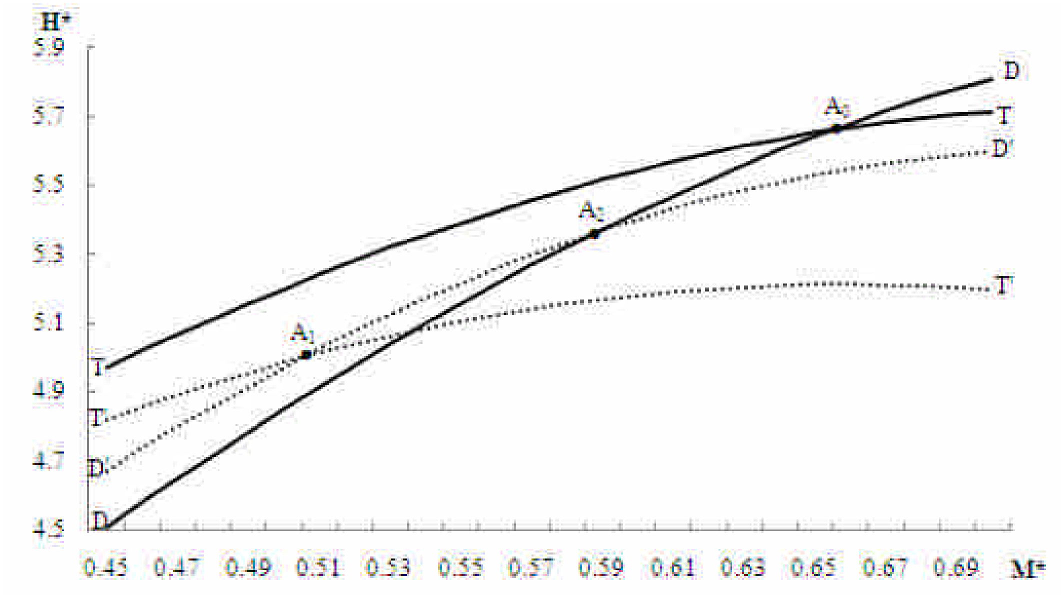

Figure 4. The impacts of greater uncertainty in housing inflation (σ increases from 12.5% to 15% per year) on the investment and financing decisions. The graph shows that greater uncertainty moves the line that characterizes the investment decision downward from line TT to $T'T'$, and moves the line that characterizes the financing decision from line DD to $D'D'$. The new equilibrium, point A_1, indicates that an investor will purchase earlier and borrow less, as compared to the initial equilibrium, point A_0.

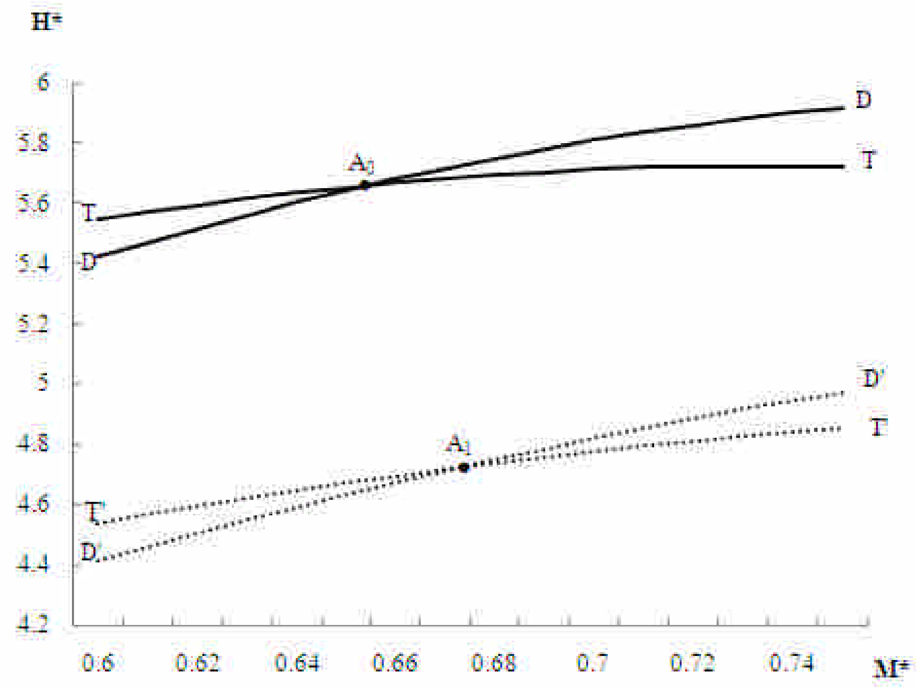

Figure 5. An increase in the tax rate, τ (from 20% to 25%), will induce an investor to accelerate purchasing and choose a higher LTV ratio. The graph shows that an increase in τ will shift line TT that characterizes choices of the investment timing downward to line $T'T'$, and line DD that characterizes choices of the LTV ratio rightward to line $D'D'$. The equilibrium point thus moves from A_0 to A_1, which indicates that an investor will purchase a property earlier and borrow more after the tax rate increases.

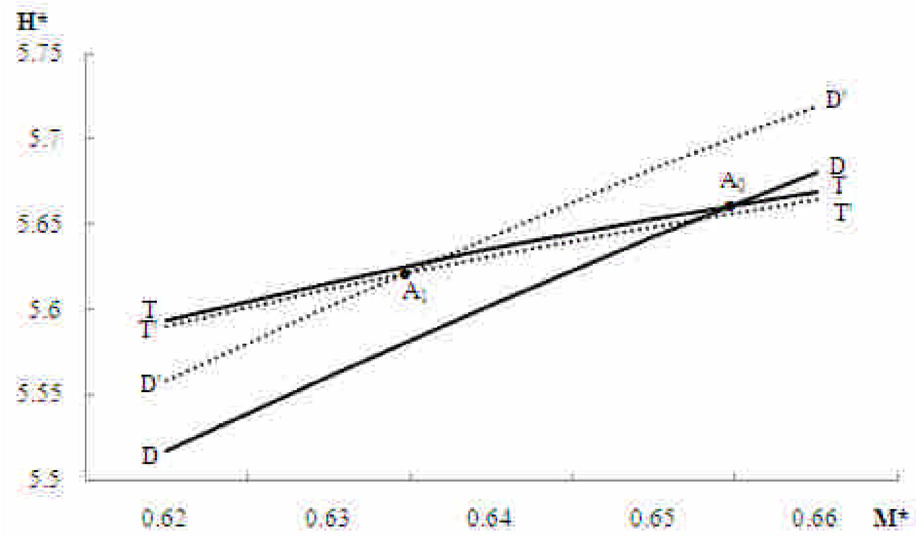

Figure 6. An increase in years allowed for depreciation (n increases from 39 years to 41 years) will induce an investor to purchase earlier and borrow less. The graph shows that an increase in n will move both line TT that characterizes choices of the investment timing downward to $T'T'$, and line DD that characterizes choices of the LTV ratio leftward to line $D'D'$. The equilibrium point thus moves from A_0 to A_1, which indicates that an investor will purchase earlier and borrow less. A decrease in the portion of depreciable capital, δ, or the elasticity of the sunk cost with respect to debt financing, ε, will exhibit similar results.

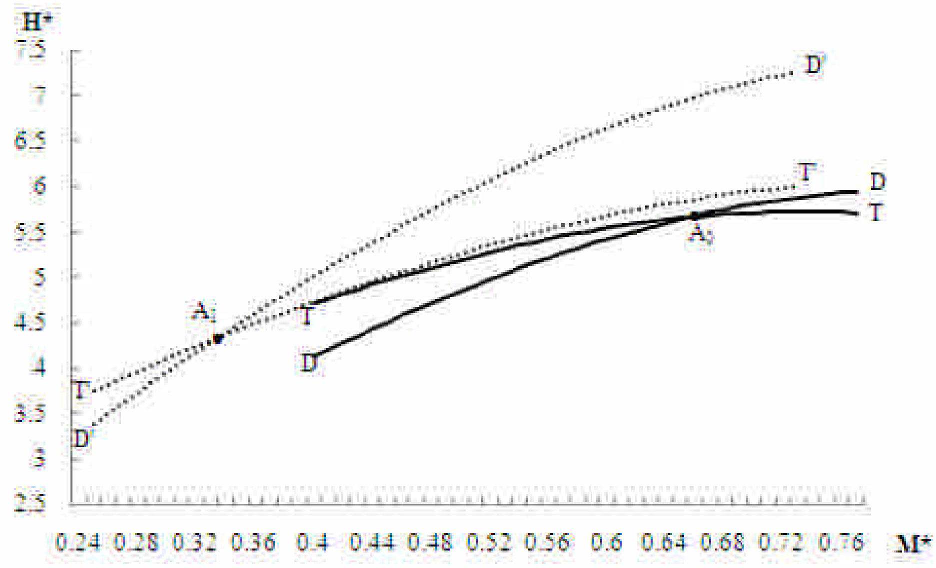

Figure 7. A decrease in the service flow rate (λ decreases from 5% to 4% per year), will induce an investor to accelerate purchasing and to choose a higher LTV ratio. The graph shows that a decrease in λ will move both line TT that characterizes choices of the investment timing upward to line $T'T'$, and line DD that characterizes choices of the LTV ratio leftward to line $D'D'$. The equilibrium point thus moves from A_0 to A_1, which indicates that an investor will accelerate the purchase and borrow less.

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