Idiosyncratic Risk and Higher-Order Cumulants: A Note*

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Abstract

We show that, when allowing for general distributions of dividend growth in a Lucas economy with multiple "trees," idiosyncratic volatility will be generically priced. In particular, we study the case when dividend growth rates follow a multivariate Normal Inverse Gaussian (NIG) distribution in some detail. The standard asset pricing results can be retrieved by assuming that growth rates follow a joint normal distribution.

Keywords: idiosyncratic risk, idiosyncratic volatility, risk premia, cumulants, NIG distribution

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1 Introduction

According to textbook financial theory (e.g., Brealey and Myers, 2003), exposure to idiosyncratic (asset-specific, unique) risk should not be rewarded on the market. This is because rational investors can eliminate idiosyncratic risk from their portfolios through diversification. The risk that cannot be diversified away is termed *systematic risk*. In order for risk-averse investors to hold a positive supply of stocks, exposure to systematic risk would have to be rewarded.

Although this story seems convincing, recent empirical research has presented evidence that idiosyncratic risk does affect risk premia, but it might not be in the direction that one would first expect. Counter intuitively, Ang, Hodrick, Xing, and Zhang (2006, 2009) find a negative relation between lagged idiosyncratic volatility and returns. They write that their results represent "a substantive puzzle" (Ang, Hodrick, Xing, and Zhang, 2006, p. 262). However, Fu (2009) argues that "The lagged idiosyncratic volatility might not be a good measure of expected idiosyncratic volatility" (Fu, 2009, p. 25). Instead using EGARCH to capture the time-varying features of idiosyncratic risk, he finds a *positive* relation between conditional idiosyncratic volatilities and returns.

This paper provides a theoretical link between idiosyncratic volatilities and expected returns. Employing an exchange-only Lucas (1978) economy in which we allow for general distributions of dividend growth rates, we find that it will appear as though idiosyncratic risk is priced in equilibrium. That is, the risk premium of a particular stock will depend not only on the covariance with aggregate consumption, but also on other quantities, including the idiosyncratic volatility of the stock and the variance of aggregate consumption. This stands in contrast with the classical result in consumption-based models that an asset's risk premium is determined only by agents' risk aversion and the covariance with aggregate consumption (e.g., Breeden, 1979).

From a technical point of view, our paper is related to Martin (2009, 2010), who also expresses equilibrium quantities in terms of cumulant generating functions. Martin (2010) considers a Lucas economy with a single risky asset (*tree*), whereas Martin (2009) considers the case of multiple risky assets. Lillestøl (1998) explores the possibility of using the multivariate NIG distribution within the areas of portfolio choice and risk analysis, and he also briefly considers equilibrium conditions assuming constant absolute risk aversion (CARA) utility. However, none of these works specifically addresses the relation between risk premia and idiosyncratic volatilities.

The remainder of the paper is organized as follows. Section 2 presents our model. In Section 3, we present our theoretical results and, finally, Section 4 concludes the paper.

2 Model

We consider an exchange-only Lucas (1978) economy in which there are n risky assets and one risk-free asset. The future dividends from assets 1 through $(n-1)$ are given by

$$D_i = D_{0i}e^{g_i}, i = 1, 2, \dots, (n-1) \tag{1}$$

and the future aggregate dividend is $D_A = D_{0A}e^{g_A}$. Here, $g_1, g_2, \dots, g_{n-1}$ and g_A are dividend growth rates, and the vector of growth rates, $(g_1, g_2, \dots, g_{n-1}, g_A)$, follows a joint probability distribution. The dividend/cash-flow from the nth asset is the difference between the aggregate dividend and the sum of the first $(n-1)$ dividends, i.e., $D_n = D_A - \sum_{i=1}^{n-1} D_i$.

There are N agents, having constant relative risk aversion: Their utility of consumption is given by

$$u(C) = \frac{C^{1-\gamma} - 1}{1-\gamma}, \tag{2}$$

where $\gamma > 1$. They maximize their expected utility of current and future consumption: They seek to maximize

$$u(C_0^j) + \beta E\left[u\left(\widetilde{C}^j\right)\right], \quad j = 1, 2, \ldots, N, \tag{3}$$

where C_0^j denotes agent j's current consumption, β is a time-preference parameter and $\widetilde{C}^j$ denotes his final consumption. All agents share the same beliefs and have access to the same information. Each of the N agents is endowed with $1/N$ shares of each risky asset, and this constitutes each agent's sole endowment.

3 Results

Since agents are homogeneous with respect to preferences, beliefs and endowments, it follows from Rubinstein's Aggregation Theorem (Rubinstein, 1974, p. 232) that we can solve for equilibrium prices by considering a representative consumer with an endowment equal to the average aggregate endowment,

$$S_i = E\left[\frac{\beta u'\left(D_A/N\right)}{u'\left(D_{0A}/N\right)} D_i\right], \quad i = 1, 2, \ldots, (n-1) \tag{4}$$

and

$$B = E\left[\frac{\beta u'\left(D_A/N\right)}{u'\left(D_{0A}/N\right)}\right], \tag{5}$$

where S_i is the price of stock i and B is the price of the bond.

In particular, under the assumed CRRA preferences, the number of agents (N) will cancel out, and we can express the asset prices as

$$S_i = \beta D_{0i} E[e^{g_i - \gamma g_A}] = \beta D_{0i} M_{[g_i - \gamma g_A]}(1), i = 1, 2, \ldots, (n-1) \tag{6}$$

and

$$B = \beta E[e^{-\gamma g_A}] = \beta M_{-\gamma g_A}(1), \tag{7}$$

where M_X denotes the moment-generating function for the random variable X.

Thus, the expected gross return on stock i is given by

$$E[(1+R_i)] = E\left[\frac{D_i}{S_i}\right] = \frac{D_{0i} E[e^{g_i}]}{\beta D_{0i} M_{[g_i - \gamma g_A]}(1)} = \frac{M_{g_i}(1)}{\beta M_{[g_i - \gamma g_A]}(1)}, \tag{8}$$

and the risk-free rate is

$$1 + R_f = \frac{1}{B} = \frac{1}{\beta M_{-\gamma g_A}(1)}. \tag{9}$$

The difference (in logs) between the two is given by

$$rp_i \equiv \ln(E[(1+R_i)]) - \ln(1+R_f) = k_{g_i}(1) + k_{-\gamma g_A}(1) - k_{[g_i - \gamma g_A]}(1), \tag{10}$$

where k is the *cumulant-generating function*, defined by $k_X(t) \equiv \ln(M_X(t))$. We call rp_i the *risk premium of asset* i.

If the cumulant-generating function exists in an open interval containing 0, then it is infinitely differentiable in this interval and thus, making a Taylor expansion around 0, we can write the cumulant-generating function of the random variable X as

$$k_X(t) = \sum_{m=1}^{\infty} \kappa_{m,X} \frac{t^m}{m!}, \tag{11}$$

where $\kappa_{m,X} \equiv k_X^{(m)}(0)$ is referred to as the *cumulant.*

Since $k_X(t) \equiv \ln M_X(t)$, there is an obvious relation between cumulants and moments. For example, the first four cumulants are

$$\kappa_{1,X} \equiv k'_X(0) = E[X], \tag{12}$$

$$\kappa_{2,X} \equiv k''_X(0) = \mathrm{Var}[X], \tag{13}$$

$$\kappa_{3,X} \equiv k_X^{(3)}(0) = \mathrm{Skew}[X], \tag{14}$$

$$\kappa_{4,X} \equiv k_X^{(4)}(0) = \mathrm{Kurt}[X] - 3\,\mathrm{Var}[X]^2, \tag{15}$$

where $\mathrm{Skew}[X] \equiv E[(X - E[X])^3]$ is the third central moment (which we call *skewness*) and $\mathrm{Kurt}[X] \equiv E[(X - E[X])^4]$ is the fourth central moment (which we call *kurtosis*).

It is also possible to define cumulant-generating functions and cumulants for multivariate random variables. In the bivariate case, one can define a cumulant-generating function $k_{(X,Y)}(t_1, t_2) \equiv \ln M_{(X,Y)}(t_1, t_2)$ with joint cumulants

$$\kappa_{(m,n),(X,Y)} \equiv \frac{\partial^m \partial^n k_{(X,Y)}}{\partial t_1^m \partial t_2^n}(0,0). \tag{16}$$

Further, it can be shown that, if $Z \equiv a_1 X + a_2 Y$, where a_1 and a_2 are constants, then

$$\kappa_{m,Z} = \sum_{j=0}^{m} \binom{m}{j} a_1^{m-j} a_2^j \kappa_{(m-j,j),(X,Y)} \tag{17}$$

(McCullagh, 1987).

Hence, the risk premium of asset i can be written as

$$\begin{aligned} rp_i &= \gamma \operatorname{Cov}(g_i, g_A) + \sum_{m=3}^{\infty} \frac{1}{m!} \left(\kappa_{m,g_i} + \gamma^m \kappa_{m,g_A} - \kappa_{m,(g_i - \gamma g_A)} \right) \\ &= \gamma \operatorname{Cov}(g_i, g_A) \\ &\quad + \sum_{m=3}^{\infty} \frac{1}{m!} \left((\gamma^m + (-\gamma)^m) \kappa_{m,g_A} + \sum_{j=1}^{m-1} \binom{m}{j} (-\gamma)^j \kappa_{((m-j),j),(g_A,g_i)} \right). \end{aligned} \tag{18}$$

If the vector $(g_1, g_2, \ldots, g_{n-1}, g_A)$ follows a joint normal distribution, then g_i, g_A, and $(g_i - \gamma g_A)$ are normally distributed. It is well known that, for a normally distributed random variable, the cumulants of order three and higher are zero. Thus, in the case when $(g_1, g_2, \ldots, g_{n-1}, g_A)$ follows a joint normal distribution, we obtain the familiar expression

$$rp_i = \gamma \operatorname{Cov}(g_i, g_A), \tag{19}$$

where $\operatorname{Cov}(g_i, g_A)$ is said to capture systematic risk. However, the normal distribution is the only distribution with a finite number of nonzero cumulants (Marcinkiewicz, 1938). Thus, unless higher-order terms cancel out in (18), the result in (19) does not hold in general.[1] Below, we discuss the case in which $(g_1, g_2, \ldots, g_{n-1}, g_A)$ follows a multivariate

[1] A well-known result is that CAPM holds when returns follow an elliptical distribution (Owen and Rabinovitch, 1983; Ingersoll, 1987). Indeed, it follows from the analysis in Hamada and Valdez (2008) that, if we let $D_i = D_{0i}(1 + \bar{g}_i)$, $i = 1, 2, \ldots, n$, where $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n)$ follows a joint elliptical distribution, then CAPM would hold. However, in order to avoid negative consumption, we model the log of dividend growth. Of course, the circumstance that a random variable is log-elliptically distributed does not imply that it is elliptically distributed (e.g., the log-normal distribution does not belong to the elliptic class). In particular, assuming that $(g_1, g_2, \ldots, g_{n-1}, g_A)$ follows a Laplace distribution (which is elliptical) and using the results in (8) and (9), we get some additional terms compared to (19). Interestingly, we obtain a

Normal Inverse Gaussian (NIG) distribution and we find that the relation in (19) does not hold in this case.

For a more general heuristic discussion, we can focus on the first two additional terms in the infinite series in (18):

$$\begin{aligned} rp_i &= \gamma\,\mathrm{Cov}(g_i, g_A) + \frac{1}{2}\left(\gamma^2 \kappa_{(1,2),(g_A,g_i)} - \gamma \kappa_{(2,1),(g_A,g_i)}\right) \\ &\quad + \frac{1}{12}\left(\gamma^4 \kappa_{4,g_A} - 2\gamma \kappa_{(3,1),(g_A,g_i)} + 3\gamma^2 \kappa_{(2,2),(g_A,g_i)} - 2\gamma^3 \kappa_{(1,3),(g_A,g_i)}\right) \\ &\quad + \text{higher order terms.} \end{aligned} \tag{20}$$

In the above expression, the second term can be written as[2]

$$\text{second term} = \frac{1}{2}\left(\gamma^2 \left(\mathrm{Cov}(g_i^2, g_A) - 2\mu_{g_i}\,\mathrm{Cov}(g_i, g_A)\right)\right. \tag{21}$$

$$\left. - \gamma\left(\mathrm{Cov}(g_i, g_A^2) - 2\mu_{g_A}\,\mathrm{Cov}(g_i, g_A)\right)\right), \tag{22}$$

while the third term can be written as

$$\begin{aligned} \text{third term} &= \frac{1}{12}\left(\gamma^4\left(Kurt[g_A] - 3\,\mathrm{Var}[g_A]^2\right)\right. \\ &- 2\gamma\left(3\left(\mu_{g_A}^2 - \mathrm{Var}[g_A]\right)\mathrm{Cov}(g_i, g_A) + \mathrm{Cov}(g_i, g_A^3) - 3\mu_{g_A}\,\mathrm{Cov}(g_i, g_A^2)\right) \\ &+ 3\gamma^2\left(4\mu_{g_i}\mu_{g_A}\mathrm{Cov}(g_i, g_A) + \mathrm{Cov}(g_i^2, g_A^2) - 2\mathrm{Cov}(g_i, g_A)^2 - 2\mu_{g_A}\mathrm{Cov}(g_i^2, g_A) - 2\mu_{g_i}\mathrm{Cov}(g_i, g_A^2)\right) \\ &\left. - 2\gamma^3\left(3\left(\mu_{g_i}^2 - \mathrm{Var}[g_i]\right)\mathrm{Cov}(g_i, g_A) + \mathrm{Cov}(g_i^3, g_A) - 3\mu_{g_i}\,\mathrm{Cov}(g_i^2, g_A)\right)\right). \end{aligned} \tag{23}$$

consumption CAPM result for continuously compounded returns assuming log-normal growth rates (19) even though the log-normal distribution does not belong to the elliptic class. This result is in line with CAPM results in continuous time (e.g., Breeden, 1979).

[2]Here, and also later, when we reformulate the third term, we use the CumulantToCentral and CentralToRaw functions in the Mathematica add-on mathStatica.

The second term depends on the expected (log) dividend growth rate of the individual stock and the expected (log) growth rate of the aggregate dividend, while the third term depends on the variances of the (log) aggregate and individual dividend growth rates. Thus, looking at the third term and using the common interpretations, it appears as though idiosyncratic risk is priced. Intuitively, investors with preferences for, for example, positive skewness would be willing to sacrifice some mean-variance efficiency for exposure to assets exhibiting positive skewness. Equation (23) suggests that preferences for higher-order moments can result in the risk premium being sensitive to idiosyncratic volatility. In addition, this equation tells us that the direction of the effect of idiosyncratic volatility on the third term in the expression for the risk premium of stock i depends on the covariance between its dividend growth rate and the growth rate of the aggregate endowment. Provided that the stock pays off well in states with low aggregate consumption, the representative agent is prepared to accept a negative risk premium, and the more volatile the stock, the lower the risk premium he is prepared to accept.

Example: Multivariate NIG distribution

In the case when $(g_1, g_2, \ldots, g_{n-1}, g_A)$ is distributed according to a multivariate NIG distribution, it follows from, for example, Lillestøl's (1998, p. 8) expression for the moment-generating function that the risk premium in (10) is exactly equal to

$$\begin{aligned} rp_i \quad = \quad & \delta\left(\sqrt{\alpha^2 - \mathbf{h}'\Phi\mathbf{h}} + \sqrt{\alpha^2 - (\mathbf{h}+\mathbf{o}_{iA})'\Phi(\mathbf{h}+\mathbf{o}_{iA})}\right. \\ & \left. - \sqrt{\alpha^2 - (\mathbf{h}+\mathbf{o}_{A})'\Phi(\mathbf{h}+\mathbf{o}_{A})} - \sqrt{\alpha^2 - (\mathbf{h}+\mathbf{o}_{i})'\Phi(\mathbf{h}+\mathbf{o}_{i})}\right), \qquad (24) \end{aligned}$$

where δ is a scale parameter, Φ is an $n \times n$ matrix related to covariance, α is a parameter controlling tail thickness, and $\mathbf{h}$ is an $n \times 1$ vector controlling the asymmetry of the distribution. Further, $\mathbf{o}_{iA}$ is an $n \times 1$ vector with 1 in its ith entry and $-\gamma$ in its nth entry and zeros in all other entries, $\mathbf{o}_A$ is an $n \times 1$ vector with $-\gamma$ as its nth entry and zeros in all other entries, and $\mathbf{o}_i$ is an $n \times 1$ vector with 1 as its ith entry and zeros in all other entries.

Now, in order to gain some intuition, consider the symmetric case in which $\mathbf{h} = \mathbf{0}$. In this case,

$$rp_i = \delta \left(\alpha + \sqrt{\alpha^2 - \phi_{ii} + \gamma(\phi_{iA} + \phi_{Ai}) - \gamma^2 \phi_{AA}} - \sqrt{\alpha^2 - \gamma^2 \phi_{AA}} - \sqrt{\alpha^2 - \phi_{ii}} \right). \quad (25)$$

Given that $\mathbf{h} = \mathbf{0}$, the variance–covariance matrix is $\Sigma = \frac{\delta}{\alpha}\Phi$, so we can rewrite the above equation as

$$rp_i = \delta \left(\alpha + \sqrt{\alpha^2 - \frac{\alpha}{\delta}(\sigma_i^2 + \gamma^2 \sigma_A^2 - 2\gamma\sigma_{iA})} - \sqrt{\alpha^2 - \frac{\alpha}{\delta}\gamma^2 \sigma_A^2} - \sqrt{\alpha^2 - \frac{\alpha}{\delta}\sigma_i^2} \right), \quad (26)$$

where $\sigma_i^2 = \mathrm{Var}[g_i]$, $\sigma_A^2 = \mathrm{Var}[g_A]$ and $\sigma_{iA} = \mathrm{Cov}(g_i, g_A)$. That is, the risk premium of an arbitrary asset is affected not only by its systematic risk (as measured by σ_{iA}), but also by its idiosyncratic volatility (σ_i) and the volatility of aggregate consumption (σ_A).

4 Conclusions

In this paper, we demonstrate that, allowing for general distributions of dividend growth rates in a Lucas economy with multiple trees, idiosyncratic volatility will generically be priced. Thus, we provide a theoretical link between expected returns and idiosyncratic volatilities.

It would be interesting to estimate our model to see how large an effect it is able to generate. In order to estimate the model, one could assume some joint distribution for growth rates (e.g., multivariate NIG), and then use some suitable estimation procedure.

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